Time reversal of free Stochastic Differential Equations and applications of non-microstates free entropy to von Neumann algebras .

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Overview

1. Summary of applications to non-microstates free entropy
    - Reminder on microstates free entropy and its applications to von Neumann algebras
    - Reminder on non-microstates free entropy and applications
    - New applications and motivation
2. Time reversal of free diffusions.
    - Background on the classical case.
    - Reversed free Brownian Motion and SDEs.
    - An application in free Probability.
    - Alternative formulas and bimodular consequences.
3. Ideas of Proofs of our applications : "weakly coarse and mixing Wasserstein rigidity".

1.1 Voiculescu's microstates Free Entropy

- $\mathcal{S}_R^n$ tracial states on the universal C^* free product $C([-R,R])^{*n} \supset \mathbb{C}\langle X_1,...,X_n\rangle$ non-commutative polynomials
- Basis of *-weak topology

$$V_{\epsilon,K}(\tau) = \{\sigma \in \mathcal{S}_R^n |\ \forall m \text{ monomials of degree less than } K \\ |\tau(m(X_1,\ldots,X_n)) - \sigma(m(X_1,\ldots,X_n))| < \epsilon\}$$

- For an n-tuple of hermitian matrices $M = (M_1,\ldots,M_n) \in (H_N^R)^n$ (i.e. $||M_i|| \le R$) one gets $\tau_M \in \mathcal{S}_R^n$:

$$\tau_M(P) = \frac{1}{N} Tr(P(M_1,...,M_n)), \quad \forall P \in \mathbb{C}\langle X_1,...,X_n\rangle.$$

- $\Gamma_R(\tau,\epsilon,K,N) = \{M \in (H_N^R)^n \mid \tau_M \in V_{\epsilon,K}(\tau)\}$.
- **Microstates free Entropy** : $\tau \in \mathcal{S}_R^n$

$$\chi_R(\tau) = \lim_{K\to\infty,\epsilon\to 0} \limsup_{N\to\infty} \left(\frac{1}{N^2} \log\left(\text{Leb}(\Gamma_R(\tau,\epsilon,K,N))\right) + \frac{n}{2}\log N \right)$$

- $(X_1, ..., X_n)$ s.a. in $(M = W^*(X_1, ..., X_n), \tau)$
  $\chi(X_1, ..., X_n) = \chi_R(\tau) > -\infty$ implies properties of M :
- M does not have property Γ (Voiculescu) : i.e. every sequence Z_m, $||Z_m||_M \leq C$, such that $||[Z_m, X]||_2 \to 0$ $\forall X \in M$ is trivial, i.e. $||Z_m - \tau(Z_m)||_2 \to 0$. (Especially, M non-amenable factor.)
- M is prime (Ge) : M is not a tensor product $M \simeq A \otimes B$ of two II_1 factors A, B.
- M has no Cartan subalgebra (Voiculescu) : There is no maximal abelian subalgebra $A \subset M$ such that its normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M), uAu^* \subset A\}$ generates M: $(\mathcal{N}_M(A))'' = M$.
- not thin (Ge, Popa) etc.

- Goal: extend those applications to non-microstates free entropy (relative to B or free mutual information) using progresses in Popa's Deformation/Rigidity Theory (that provided alternative proofs and much more for free Groups factors $L(\mathbb{F}_n)$ recently, e.g. strong solidity).

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- In general, it is hard to check $\chi(X_1, ..., X_n) > \infty$ (or even $\delta_0(X_1, ..., X_n) > 1$).
- **Non-microstates free entropy** χ^* : Alternative formula using free stochastic differential equations and free Fisher's information and expected to be equal ($\chi \leq \chi^*$ known by a result of [Biane-Capitaine-Guionnet], and equality if $n = 1$).

- Reminder of definition : Start by considering $X_1, \ldots, X_n \in (M = W^*(X_1, \ldots, X_n), \tau)$ finite von Neumann algebra, $X_1, \ldots, X_n$ algebraically free self-adjoints and define the **free difference quotient** $\partial_i : \mathcal{C} = \mathbb{C}\langle X_1, \ldots, X_n \rangle \to \mathcal{C} \otimes \mathcal{C}$ the unique derivation with :

$$\partial_i(X_j) = 1 \otimes 1\delta_{i=j}$$

  Look at $\partial_i : L^2(M, \tau) \to L^2(M, \tau) \otimes L^2(M, \tau)$
- Define $\xi_i = \partial_i^* 1 \otimes 1 \in L^2(M, \tau)$ **conjugate variables**, if they exist. This is the free analogue of the score function.
- **Free Fisher information** is defined as ∞ if they don't exist and otherwise:

$$\Phi^*(X_1, \ldots, X_n) = \sum_{i=1}^{n} ||\xi_i||_2^2.$$

- Consider

$$X_{i,t} = X_{i,0} + S_{i,t},$$

$S_{i,t}$ free Brownian motion, and $\xi_{i,t}$ conjugate variables for $X_{1,t}, \dots, X_{n,t}$, then non-microstates **free entropy** is defined as :

$$\chi^*(X_1, \dots, X_n) = \frac{1}{2}\int_0^\infty \left(\frac{n}{1+t} - \Phi^*(X_{1,t}, \dots, X_{n,t})\right) dt + \frac{n}{2}\log(2\pi e),$$

- Explication for this formula (or its Orstein-Uhlenbeck variant): "relative entropy of the process considered backwards in time and using Girsanov formula for the density".

1.2 Known applications of non-microstates free entropy and free Fisher information

- [D2008] If $\chi^*(X_1, ..., X_n) > -\infty$, $W^*(X_1, ..., X_n)$ is a factor.
- [D2008]If $\Phi^*(X_1, ..., X_n) < \infty$, $W^*(X_1, ..., X_n)$ doesn't have property Γ.
- Recent results in a joint work with Adrian Ioana :

Theorem (Ioana, D. 2012)

Let $(M = W^*(X_1, ..., X_n), \tau)$. Assume that either

- $\Phi^*(X_1, ..., X_n) < \infty$ *and* $n \geqslant 3$, *or*
- $\xi_i = \partial_i^*(1 \otimes 1), \partial_i^*(1 \otimes \xi_i)$ *exists and belongs to* M, $\forall i \in \{1, ..., n\}$, $n \geq 2$.

Then, M is prime and does not have property Γ.
Actually M is a non-L^2-rigid II_1 factor in the sense of Jesse Peterson (which implies M is prime and does not have property Γ [Peterson 2006]).

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- The main example of $X_1, \dots, X_n$, with $\xi_i = \partial_i^*(1 \otimes 1), \partial_i^*(1 \otimes \xi_i) \in M$ is $X_i = Y_i + S_{i,t}$, Y_i and $S_{i,t}$ free. In this case, we can conclude more (without assuming $X_1, \dots, X_n$ R^ω-embeddable as for the corresponding microstate free entropy result):

Theorem (Ioana, D. 2012)

Let (M, τ) be a tracial von Neumann algebra and $X_1, \dots, X_n \in M$ be $n \geqslant 2$ self-adjoint elements. Let $\{S_1, \dots, S_n\} \in L(\mathbb{F}_n)$ be the canonical semicircular family and $\varepsilon > 0$. Denote by $M_\varepsilon \subset M * L(\mathbb{F}_n)$ the von Neumann subalgebra generated by $X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n$. Then M_ε is a non-L^2-rigid II_1 factor that does not have a Cartan subalgebra.
(The conclusion also holds for $S_1, \dots, S_n$ replaced by $Y_1, \dots, Y_n$ with $\xi_i = \partial_{Y_i}^*(1 \otimes 1), \partial_{Y_i}^*(1 \otimes \xi_i) \in M$ and free from $X_1, \dots, X_n \notin \mathbb{C}$)

1.2 Applications of non-microstates free entropy.

Theorem (Ioana, D. 2012)

Let $(M = W^*(X_1, \dots, X_n), \tau)$ and $\partial_i : M \to L^2(M)\bar{\otimes}L^2(M)$ the free difference quotient. Assume $\xi_i = \partial_i^*(1 \otimes 1)$ exists, $\xi_i \in D(\bar{\partial})$ and $\bar{\partial}_i(\xi_j) \in (M\bar{\otimes}M^{op}) \subset L^2(M\bar{\otimes}M^{op}) \cong L^2(M)\bar{\otimes}L^2(M)$ (Lipschitz conjugate variable), for all $1 \leqslant i, j \leqslant n$.
Then M is a II_1 factor which does not have a Cartan subalgebra. Moreover, $M\bar{\otimes}Q$ does not have a Cartan subalgebra, for any II_1 factor Q.

- The assumption Lipschitz conjugate variable is the one under which [D.2010] shows (using ideas of [Shlyakhtenko-2007]) that if, moreover, M is R^ω embeddable, then $\delta_0(X_1, \dots, X_n) = n$.
- The result $M \otimes Q$ has no Cartan subalgebra is not known by microstates free entropy techniques, but known for $M = L(\mathbb{F}_n)$ by [Popa-Ozawa 2007,Popa-Vaes 2011].

1.2 Reminder on non-microstates mutual information

- Let $A_1, \dots, A_n \subset (M = W^*(A_1, \dots, A_n), \tau)$ be algebraically free subalgebras and define the unique derivation $\delta_i : A = Alg(A_1, \dots, A_n) \to A \otimes A$:

$$\delta_i(a_j) = (a_j \otimes 1 - 1 \otimes a_j)\delta_{i=j}, a_j \in A_j.$$

Look at $\delta_i : L^2(M, \tau) \to L^2(M, \tau) \otimes L^2(M, \tau)$

- Define $\mathcal{J}_i = \delta_i^* 1 \otimes 1 \in L^2(M, \tau)$ the **liberation gradients**, if they exist and :

$$\varphi^*(A_1, \dots, A_n) = \sum_{i=1}^{n} ||\mathcal{J}_i||_2^2.$$

- Using $U_{i,t}$, n free unitary Brownian motions, i.e. solving the SDE : $U_{i,t} = 1 - \frac{1}{2}\int_0^t U_{i,s} ds + i \int_0^t dS_{i,s} U_{i,s}$, and using the liberation process $(U_{i,s} A_i U_{i,s}^*)$,we define mutual information

$$i^*(A_1, \dots, A_n) = \frac{1}{2}\int_0^\infty \varphi^*(U_{1,s} A_1 U_{1,s}^*, \dots, U_{n,s} A_n U_{n,s}^*) ds.$$

1.2 Applications of non-microstates mutual information.

Theorem (Ioana, D. 2012)

Let $(M = Wj(A_1, ..., A_n), \tau)$ tracial $n \geqslant 2$ generated $A_1, ..., A_n \neq \mathbb{C}1$ with $\varphi^*(A_1; ...; A_n) < \infty$ such that A_1 is diffuse, and A_2 is a non-amenable II_1 factor.
Then M is a non L^2-rigid II_1 factor. Thus, M is prime, does not have property Γ nor property (T).

Theorem (Ioana, D. 2012)

Let $A_1, ..., A_n \subset (M_1, \tau_1)$ be diffuse von Neumann subalgebras free from $u_1, ..., u_n$ unitary elements, for some $n \geqslant 2$. Denote by $N = W^*(u_1 A_1 u_1^*, ..., u_n A_n u_n^*)$.
Assume that A_1 is a non-amenable II_1 factor and that $u_2 \notin \mathbb{C}u_1$.
Then N does not have a Cartan subalgebra.

This complements results of [Hiai, Miyamoto,Ueda] by microstates techniques when $A_1, ..., A_n$ are amenable.

1.2 Ideas behind recent Applications of non-microstates free entropy and mutual information.

- Build $\alpha_t : M \to \tilde{M} \supset M$ deformations, i.e. trace preserving $*$-homomorphisms with $||\alpha_t(x) - x||_2 \to_{t\to 0} 0$ solving a free SDE and use Popa's Deformation/Rigidity Theory (mainly spectral gap rigidity)
- If $\tilde{M} = M * L(\mathbb{F}_\infty)$ (e.g. in the case of Lipschitz conjugate variable [D2010]) one can use [Ioana 2012] to prove absence of Cartan subalgebras (first force $\alpha_t(M) \prec_{\tilde{M}} M$ and get a contradiction with some non-amenability in M)
- If $L^2(\tilde{M}) \ominus L^2(M)$ is a direct sum of coarse $M - M$ bimodule $L^2(M) \otimes L^2(M)$ (or weaker, weakly contained in the coarse and mixing), then one can use idea's of [Peterson 2006] and get primness results.
- Pb: Obtaining those dilations is really hard and require at least finite Fisher information (or closable derivations), a finite entropy assumption seems out of reach.

- New idea : look at non-trace preserving $*$-homomorphism (e.g. $X = X_0 \mapsto X_t = X_0 + S_t$ as in definition of free entropy) and exploit the flip homomorphism in $W^*(X_0, S_t) *_{W^*(X_t)} W^*(X_0, S_t)$.
- New problem : Control $W^*(X_t) \subset W^*(X_0, S_t)$. One expects ideally $W^*(X_0, S_t) = W^*(X_t) * L(\mathbb{F}_\infty)$ to exploit [Ioana 2012] and get absence of Cartan subalgebra results.
- But at this stage, one can only control $L^2(W^*(X_0, S_t)) \ominus L^2(W^*(X_t))$ as $W^*(X_t)$-bimodule and use idea's of [Peterson 2006] to get primeness results.

1.3 New Non-Γ Results

Define following [Connes-Shlyakhtenko]:

$$\delta^\star(X_1, ..., X_n) = n - \limsup_{t \to 0} t\Phi^*(X_1 + S_{1,t}, ..., X_n + S_{n,t}).$$

Theorem

For any $Z_m \in W^*(X_1, ..., X_n)$ with $||Z_m|| \leq 1$ for all m, and $\limsup_{m\to\infty} ||[Z_m, X_i]||_2 = 0$, i=1,...,n. then if $\delta^\star(X_1, ..., X_n)$ is close to n :

$$\limsup_{m\to\infty} ||Z_m - \tau(Z_m))||_2 \leq 46(\frac{n - \delta^*(X_1, ..., X_n)}{n-1})^{1/8}$$

Especially, if $\delta^\star(X_1, ..., X_n) = n$ (e.g. if $\chi^*(X_1, ..., X_n) > -\infty$), then $W^*(X_1, ..., X_n)$ is a factor without property Γ.
Moreover if $W^*(X_1, ..., X_n)$ is a factor and $\delta^*(X_1, ..., X_n)$ is close to n, then $W^*(X_1, ..., X_n)$ does not have property Γ.

One can also prove that if

$$\liminf_{t\to 0} t \sup_i \left(\|\partial^*_{X_{i,t}} 1 \otimes 1)\|^2 + \|\partial^*_{X_{i,t}} (1 \otimes \partial^*_{X_{i,t}} 1 \otimes 1))\| \right) = 0,$$

then $X_1, \ldots, X_n$ is a non-Γ set for M in the sense of [Peterson 2004], i.e., $\exists c > 0 \forall Z \in L^2(M)$:

$$\|Z - \tau(Z)\|_2 \leq c \sum_{i=1}^{n} \|[Z, X_i]\|_2.$$

Theorem

Assume $\delta^\star(X_1, ..., X_n) = n$ and $\mathbb{C}\langle X_1, ..., X_n\rangle$ contain a non-Γ set for $M = W^*(X_1, ..., X_n)$ (or even only a non-amenability set) then M is a prime II_1 factor.

Theorem

Assume $i^*(A_1, ..., A_n) < \infty$ and A_1, A_2 diffuse, A_1 non-amenable then $M = W^*(A_1, ..., A_n)$ is a prime II_1 factor without property Γ.

- Note that if we knew that $\chi^*(X_1, ..., X_n) \leq -i^*(W^*(X_1), ..., W^*(X_n)) + \sum_{i=1}^n \chi(X_i)$ this would imply primness as soon as $\chi^*(X_1, ..., X_n) > -\infty$ and $n \geq 3$.
- Knowing that $\chi_{orb}(W^*(X_1), ..., W^*(X_n)) \leq -i^*(W^*(X_1), ..., W^*(X_n))$ and $\chi(X_1, ..., X_n) = \chi_{orb}(W^*(X_1), ..., W^*(X_n)) + \sum_{i=1}^n \chi(X_i)$[HMU], one recovers Ge's result for $n \geq 3$

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2. Time reversal of free diffusions.
   - Background on the classical case.
   - Reversed free Brownian Motion and SDEs.
   - An application in Free Probability.
   - Alternative formulas and bimodular consequences.
3. Ideas of Proofs of applications : "weakly coarse and mixing Wasserstein rigidity".

Consider a solution on $[0, T]$ of a classical Markovian SDE :

$$X_t = X_0 + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB_s.$$

- Problem: When is $Y_t = X_{T-t}$ also a diffusion ? (i.e. solve the same kind of SDE)
- Original Motivation (Nelson): Model Quantum Mechanics (which is reversible) in Stochastic Mechanics. [Nelson '67] found that formally, there should be a correction of the drift by appropriate score function, i.e. Y_t should satisfy :

$$Y_t = Y_0 + \int_0^t \overline{b}(T-s, Y(s)))ds + \int_0^t \sigma(T-s, X(s))d\overline{B}_s,$$

with the new drift :

$$\overline{b}_j(T-s, y) = \frac{\sum_i \nabla_i((\sigma\sigma^*)_{ji}p_s)}{p_s}(y) - b_j(T-s, y),$$

where p_t is the density with respect to Lebesgue measure of $(Y_{1,t}, \dots, Y_{n,t})$.

2.1 Background on Classical Time reversal

Consider a solution on $[0, T]$ of a classical Markovian SDE :

$$X_t = X_0 + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB_s.$$

- Mathematical Work of [Anderson '82][Föllmer '86],[Pardoux, Haussmann],[Pardoux],[Millet,Nualart,Sanz '89][Jacod]
- Definitive answer in [Millet,Nualart,Sanz '89] : The reversed process satisfy a martingale problem as soon as the formula for the reversed drift exist.
- More interesting for us, [Pardoux] gave under stronger conditions an explicit formula for the brownian motion driving the time reversed process. (Enlargement of filtration method) :

$$\overline{B}_t = B_{T-t} - B_T - \int_{T-t}^{T} \frac{\sum_i \nabla_i(\sigma_i p_s)}{p_s}(X_s)ds.$$

- One can check this is a Brownian motion by a Levy Theorem manageable in the free case

2.2 Time Reversal of free SDEs.

Consider a strong solution on $[0,T]$ of a free (Markovian) SDE :

$$X_{i,t} = X_{i,0} + \int_0^t V_i(s, X(s))ds + \int_0^t Q_i(s, X_s)\# dS_s.$$

- Using [Biane,Speicher], this can be defined and solved for regular (in $\mathcal{C} = \mathbb{C}\langle X_1, \dots, X_n\rangle$) V_i, Q_i. A strong solution means $X_t \in W^*(X_0, S_s, s \leq t)$. Recall $(a \otimes b)\# S = aSb$.
- Same Problem: When does $Y_t = X_{T-t}$ solve a free SDE ?
- Using Ito formula, one can rewrite the equation with derivations $\delta_i : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$., and $\Delta_{Q,V} : \mathcal{C} \to \mathcal{C}$, there is a $*$-homomorphism $\alpha_t(X) = X_t$ such that for any $P \in \mathcal{C}$:

$$\alpha_t(P) = P + \int_0^t ds \alpha_s(\Delta_{Q,V}(P)) + \int_0^t \alpha_s \otimes \alpha_s(\delta(P))\# dS_s.$$

More precisely, for any $P \in \mathcal{C}$, if $\tau_s = \tau \circ \alpha_s$:

$$\alpha_t(P) = P + \int_0^t ds\alpha_s(\Delta^{\tau_s}_{Q,V}(P)) + \int_0^t \alpha_s \otimes \alpha_s(\delta(P))\# dS_s.$$

- Explicitly, we have : $\delta_j(P) = \sum_i \partial_i(P)\# Q_{i,j}$, $\Delta^{\tau}_{Q,V}(P) = \sum_i \partial_i(P)\# V_i + \sum_{i,k,l} m \circ 1 \otimes \tau \otimes 1((\partial_k \otimes 1)\partial_l(P)\#(Q_{k,i} \otimes Q_{l,i}))$, We will assume $\Delta^{\tau_s}_{Q,V}$ has the form $(W_{j,s} \in \mathcal{C})$: $\Delta^{\tau_s}_{Q,V}(P) = \sum_j \delta_j(P)\# W_{j,s} + \sum_i m \circ 1 \otimes \tau_s \otimes 1(\delta_i \otimes 1\delta_i(P))$.
- We can consider $\delta_{i,s}$ defined on $Alg(X_{1,s}, ..., X_{n,s})$, for $P \in \mathcal{C}$, by

  $$\delta_{i,s}(P(X_{1,s}, ..., X_{n,s})) = (\alpha_s \otimes \alpha_s)\delta_i(P) \in L^2(W^*(X_s) \otimes W^*(X_s)).$$

  We will assume $\xi_{i,s} := \delta^*_{i,s} 1 \otimes 1$ exists $s > 0$ and is in M and supplementary assumptions, proved for liberation processes and free brownian motion by Voiculescu, and that we can also check when $Q_{ij} = 1 \otimes 1\delta_{i=j}$, $V_i = D_i V$, $V \in \mathcal{C}$.

2.2 Reversed free Brownian Motion.

Assumption (C):

1. $s \in [0, T) \mapsto \overline{\xi}_s = \xi_{T-s}$ is left continuous with right limits when seen as valued in $L^2(M)^n$.
2. $\exists C > 0, ||\overline{\xi}_{i,s}|| < C/\sqrt{T-s}, s < T$
3. $\exists D \geq 0, \alpha > 0 \forall t < s < T,$

$$||E_{W^*(X_{1,T-t},\dots,\overline{X}_{i,T-t})}(\overline{\xi}_{s,i}) - \overline{\xi}_{t,i}||_2 \leq D(s-t)^{\alpha}.$$

4. For any $P \in \mathcal{C}$, for any $s \leq T$, there exists paths $(K_t^s(P), L_t^s(P))_{t \in [0,s]} \in C^1([0,s], C^2(X_1, \dots, X_n))^2$ such that $K_s^s(P) = L_s^s(P) = P$ and for $t \leq s$

$$\frac{\partial K_t^s(P)}{\partial t} + \Delta_{Q,V}^{\tau_t}(K_t^s(P)) = 0,$$

$$\frac{\partial L_t^s(P)}{\partial t} - \Delta_{Q,V}^{\tau_{T-t}}(L_t^s(P)) = 0.$$

5. +Extra technical assumptions

2.2 Reversed free Brownian Motion.

Theorem

Under assumption (C), $\overline{S}_{i,t} := S_{i,T-t} - S_{i,T} + \int_0^t ds \overline{\xi}_{i,s}, t \in [0,T]$ *is a free brownian motion adapted to the filtration* $\overline{\mathcal{F}}_s = W^*(B, \alpha_{T-t}(P), P \in \mathcal{C}, t \in [0,s], \overline{S}_{i,t}, t \in [0,s])$.

2.2 Reversed free Brownian Motion.

- Key Idea of Proof: one uses a free Paul Levy's Thm [Biane-Capitaine-Guionnet] characterizing free Brownian motion.

Theorem (Biane-Capitaine-Guionnet)

Let B_s be an increasing filtration of von Neumann algebras in (M, τ), $Z_s = (Z_s^1, ..., Z_s^m), s \in \mathbb{R}_+$ an m-tuple of self-adjoint processes adapted to this filtration $Z_0 = 0$ and :

1. $E_{B_s}(Z_t) = Z_s$
2. $Z_t - Z_s = U_{t,s} + V_{t,s}$ *with* $\tau(|U_{t,s}|^4) \leq K(t-s)^{3/2}$ *and* $\tau(|V_{t,s}|^2) \leq K(t-s)^2$
3. $\tau(Z_t^k A Z_t^l C) = \tau(Z_s^k A Z_s^l B) + (t-s)1_{\{k=l\}}\tau(A)\tau(C) + o(t-s)$ *for any* $A, C \in B_s$.

Then Z is a free Brownian motion adapted to B_s.

2.2 Reversed free Brownian Motion.

Theorem

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- Key Idea : one uses a free Paul Levy's Thm [Biane-Capitaine-Guionnet] characterizing free Brownian motion.
- To check the martingale property, one uses the PDE solution K_t^s chosen so that:

$$\alpha_{T-s}(X) = K_0^{T-s}(X) + \int_0^{T-s} (\alpha_u \otimes \alpha_u)(\delta(K_u^{T-s}(X)))\# dS_u.$$

  This reduces the martingale property to the adjoint definition of $\xi_{i,s}$.
- The other estimates are easy.

2.2 Reversed free SDE.

Theorem

Under assumption (C), if $\overline{S}_{i,t} := S_{i,T-t} - S_{i,T} + \int_0^t ds\overline{\xi}_{i,s}$*, then for any* $P \in \mathcal{C}$ $\overline{\alpha_t}(P) := \alpha_{T-t}(P)$ *satisfy the following free SDE :*

$$\overline{\alpha}_t(X) = \overline{\alpha}_0(X) - \int_0^t ds[\overline{\alpha}_s(\Delta_{Q,V}^{\tau T-s}(X)) + \Delta_s\overline{\alpha}_s(X)] + \int_0^t \overline{\alpha}_s \otimes \overline{\alpha}_s(\delta(X))\#d\overline{S}_s,$$

where $\Delta_s = \delta^*_{T-s}\overline{\delta_{T-s}}$, $s < T$.

One mainly needs the following identity for some processes $Y_s \in W^*(\alpha_{T-s}(\mathcal{C})) \cap D(\Delta_s)$:

$$\int_u^v \overline{\delta}_s(Y_s)\#d\overline{S}_s + \int_{T-v}^{T-u} \overline{\delta}_{T-s}(Y_{T-s})\#dS_s = \int_u^v \Delta_s(Y_s)ds.$$

Actually for $Y \in D(\Delta)$ generator of the form : $\mathcal{E}(f) = \int_0^T ||\overline{\delta_s}f(s)||_2^2 ds$.

2.3 An application in free probability

- Consider the special case of liberation process of 2 projections $p, q,\ q_t = q,\ p_t = u_t p u_t^*$ with $u_t = 1 - \frac{1}{2}\int_0^t u_s ds + i\int_0^t dS_t u_t$ so that :

$$p_t = p + \int_0^t (\tau(p) - p_s) ds + i\int_0^t [dS_s, p_s]$$

- If $\tilde{p}_t = p_{T-t}$, then our result states :

$$\tilde{p}_t = \tilde{p}_0 + \int_0^t (\tau(p) - \tilde{p}_s - [\tilde{p}_s, \mathcal{J}_{i,s}]) ds + i\int_0^t [dS_s, \tilde{p}_s],$$

where $\mathcal{J}_{i,s}$ is the liberation gradient computed at time s.

2.3 An application in free probability

- At the end of [Bercovici,Collins,Dykema,Li,Timotin 2008] and in the clarification of a gap in a proof in [Collins,Kemp 2012], if $R_T = p_T \wedge q$, $\tau(p), \tau(q) \leq 1/2$, the authors are interested in computing the derivative of $F_T(s) = \tau(R_T p R_T - R_T)^2$. For $s \geq T$, forward Ito calculus applies to get the right derivative [Collins,Kemp 2012]

$$F'_{T,r}(T) = 2\tau(R_T)(1 - \tau(p)) \geq 0.$$

- Forward Ito Calculus don't say anything about the left derivative (this was the original main gap), but the backward equation does :

$$F'_{T,l}(T) = -2\tau(R_T)(1 - \tau(p)) \leq 0.$$

Thus (if I didn't make a sign mistake to get my derivative) F_T is differentiable at T only if $\tau(R_T) = 0$.

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Theorem

Assume assumption (C).

1. *For any* $P \in \mathcal{C}$ *let us write* $R_P^t(\overline{X}_u) = \alpha_{T-u}(L_u^t(X))$, *then :*

$$\overline{\alpha_t}(P) = \int_u^t \delta_v(R_P^t(\overline{X}_v))\# d\overline{S}_v + R_P^t(\overline{X}_u) - \int_u^t dv \Delta_v(R_P^t(\overline{X}_v)).$$

2. *Let us write* $Q_P^t(u) = E_{W^*(\alpha_{T-u}(\mathcal{C}))}(\alpha_{T-t}(P)), u < t$ *then* $v \mapsto 1_{[u,t)}(v) Q_P(v,t)$ *is in* $D(\mathcal{E})$ *and for any* $Z \in D(\mathcal{E}), a_u, b_u \in W^*(\overline{X}_u)$ *we have :*

$$(P_t - Q_P^t(u) - \int_u^t \overline{\delta}_s(Q_P^t(s))\# d\overline{S}_s) \perp \overline{a}_u \int_u^T \overline{\delta}_s(Z_s)\# d\overline{S}_s \overline{b}_u.$$

3. *Fix* t. *For almost all* $u \in [t, T]$. *The* $M_u = W^*(\overline{X}_u)$ *bimodule generated by* $P_t - E_u(P_t)$ *for* $P_t \in M_t$, *is weakly contained into the coarse bimodule* $L^2(M_t) \otimes L^2(M_t)$ *and mixing.*

2.4 Remarks on bimodularity properties

First recall the following :

Definition (Peterson 2006,Peterson-Sinclair 2009)

Let (M, τ) be a tracial von Neumann algebra. We say that an M-M bimodule $\mathcal{H}$ is *mixing* if for any sequence $a_n \in (M)_1$ such that $a_n \to 0$, weakly, we have

$$\sup_{x \in (M)_1} |\langle a_n \xi x, \eta \rangle| \to 0 \text{ and } \sup_{x \in (M)_1} |\langle x \xi a_n, \eta \rangle| \to 0, \text{ as } n \to \infty, \forall \xi, \eta \in \mathcal{H}.$$

2.4 Remarks on bimodularity properties

- Those bimodularity properties will be exactly what is needed for primness results starting from finite non-microstates entropy.
- Note that, since we don't know that $P_t - Q_P(u,t)$ is a stochastic integral, the second formula is not enough to prove the statement about the bimodularity property.
- The trick is to see

$$P_t - Q_P(u,t) = \int_u^t \delta_v(R_{P,t}(\overline{X}_v)) \# d\overline{S}_v - \int_u^t dv(1-E_u)\Delta_v(R_{P,t}(\overline{X}_v))$$

And use the fact that $\Delta_v = \delta_v^* \delta_v$ also make appear a coarse bimodule, as the stochastic integral (which is also an adjoint operator valued in a coarse bimodule in Malliavin calculus sense).

- For $X_t = X_0 + S_t$, the second statement is really interesting. By Voiculescu's result, the conjugate variable at time t is : $\xi_{i,t} = E_{W^*(X_t)}(\frac{1}{t}S_{i_t}) = \frac{X_t}{t} - \frac{1}{t}E_{W^*(X_t)}(X_0)$ so that $Q_{X_i}(u,T) = X_{i,u} - u\xi_{i,u}$ and thus $\int_u^T ||\delta_s(\xi_{i,s})||_2^2 ds < \infty$. Actually, the proof also gives for $t > u$:

$$||\xi_{i,u}||_2^2 \geq ||\xi_{i,T}||_2^2 + \int_u^T ||\delta_s(\xi_{i,s})||_2^2 ds.$$

If $P_t - Q_P(u,t)$ were stochastic integrals, we would have equality, proving an hold conjecture of Voiculescu about (absolute) continuity of Fisher information along free Brownian motion.

Overview

1. Summary of applications to non-microstates free entropy
   - Reminder on microstates free entropy and its applications to von Neumann algebras
   - Reminder on non-microstates free entropy and applications
   - New applications and motivation
2. Time reversal of free diffusions.
   - Background on the classical case.
   - Reversed free Brownian Motion and SDEs.
   - An application in Free Probability.
   - Alternative formulas and bimodular consequences.
3. Ideas of Proofs of our applications : "weakly coarse and mixing Wasserstein rigidity".

3 Ideas of Proofs of our applications : "weakly coarse and mixing Wasserstein rigidity".

- Instead of a dilation $\alpha_t : M = W^*(X_1, ..., X_n) \to \tilde{M}$ available only in case of Finite Fisher information (or starting with any other closable derivation), we have now only the building blocks $W^*(X, S_t) *_{W^*(X_1+S_{1,t},...X_n+S_{n,t}} W^*(X, S_t)$
We think of this as a coupling as those appearing for the definition of Wasserstein distance, and we obtained and will consider couplings with extra control on bimodularity properties.
- We need to (slightly) generalize the results of [Peterson 2006] and [Ioana D. 2012] in this setting, with finite entropy playing the role of a quantitative estimation of the way the free difference quotient can be approximated by closable derivations.

Definition

A *weakly coarse and mixing Wasserstein coupling* (wCMW coupling) of M_1 and M_2 is a von Neumann algebra (M, τ) with two trace preserving (unital) $*$ homomorphisms
$\iota_1 : (M_1, \tau_1) \to (M, \tau)$, with expectation $E_1 = E_{\iota_1(M)}$,
$\iota_2 : (M_2, \tau_2) \to (M, \tau)$ such that the submodule

$$\mathcal{K}(\iota_1, \iota_2) := \overline{Span\{\iota_1(x)(\iota_2(y) - E_1(\iota_2(y)))\iota_1(z); x, z \in M_1, y \in M_2\}}^{L^2}$$

is a mixing and weakly contained in the coarse bimodule $L^2(\iota_1(M_1)) \otimes L^2(\iota_1(M_1))$ as $\iota_1(M_1) - \iota_1(M_1)$ bimodule, and symmetric statements in changing M_1, M_2.

Lemma

If N_1 is a wCMW coupling for $M_1 - M_2$ and N_2 is a wCMW coupling for $M_2 - M_3$, then so is $N_1 *_{\iota_2(M_2)} N_2$ for $M_1 - M_3$.

3 Ideas of Proofs of our applications : "weakly coarse and mixing Wasserstein rigidity".

Definition

A densely defined derivation $\delta : D(\delta) \to \mathcal{H}$ on M is said to have a *weakly coarse and mixing Wasserstein dilation* if there exists for any $t \in (0,1)$ a weakly coarse and mixing Wasserstein coupling M_t for M and itself with embeddings ι_1^t and ι_2^t and if moreover there are $0 < c < C < \infty$ such that for any P in $D(\delta)$:

$$\limsup_{t \to 0} \frac{1}{\sqrt{t}} ||\iota_1^t(P) - E_{\iota_2^t(M)}(\iota_1^t(P))||_2 \leq C||\delta(P)||_2,$$

$$\liminf_{t \to 0} \frac{1}{\sqrt{t}} ||\iota_1^t(P) - E_{\iota_2^t(M)}(\iota_1^t(P))||_2 \geq c||\delta(P)||_2.$$

and symmetrically changing 1 and 2.

3 Ideas of Proofs of our applications : "weakly coarse and mixing Wasserstein rigidity".

- Using that

$$P(X_0) - E_{W^*(X+S_t)}P(X_0) - \delta(P)(X_t)\#\overline{S_t}$$
$$= \int_0^t \delta_v(R_P^t(\overline{X}_v) - \delta(P)(X_t))\#d\overline{S}_v - \int_0^t dv(1 - E_u)\Delta_v(R_P^t(\overline{X}_v)).$$

and $\int_0^t ||\xi_v||_2 dv \leq \sqrt{t}\sqrt{\int_0^t ||\xi_v||_2^2 dv} = o(\sqrt{t})$ when $\chi^*(X_1, ..., X_n) > -\infty$. One can see that $M_t = W^*(X, S_t) *_{W^*(X_1+S_{1,t},...X_n+S_{n,t})} W^*(X, S_t)$ gives a wCMW dilation of the free difference quotient in this case.

3 Ideas of Proofs of our applications : "weakly coarse and mixing Wasserstein rigidity".

Note that we can win some results usually given by symmetry by a free product with amalgamation trick.

Lemma

If δ has a wCMW-dilation, then it has a wCMW-dilation (α_t, β_t) such that moreover, for any $P \in D(\delta)$,

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \|\alpha_t(P) - \beta_t(P)\|_2 \geq c\|\delta(P)\|_2,$$

$$\limsup_{t\to 0} \frac{1}{\sqrt{t}} \|\alpha_t(P) - \beta_t(P)\|_2 \leq C\|\delta(P)\|_2.$$

- We need to (slightly) generalize the results of [Peterson 2006] and [Ioana D. 2012] in this setting, with finite entropy giving a quantitative estimate on the way the free difference quotient can be approximated by closable derivations.

3 A variant of Peterson L^2-Rigidity : "weakly coarse and mixing Wasserstein rigidity".

Definition

An inclusion of finite von Neumann algebras $(Q \subset M, \tau)$ is said to be wCMW-rigid if for any densely defined derivation $\delta : D(\delta) \to \mathcal{H}$ having a wMCW-dilation there is (maybe) another wCMW-dilation such that $\sup_{x \in (Q)_1} \|\iota_1^t(x) - \iota_2^t(x)\|_2 \to_{t \to 0} 0$ ($(Q)_1$ unit ball of Q).

Theorem (Variant of Peterson 2006)

If N is a non-amenable II_1 factor which is non-prime or has property Γ then $N \subset N$ is wCMW-rigid.

Theorem (Variant of Ioana-D 2012)

Let M be a II_1 factor. Assume that there exists an unbounded derivation $\delta : M_0 \to \mathcal{H}$ having a wCMW-dilation relative to B such that M_0 contains a non-Γ set.
Then M is not wCMW-rigid. Thus, M is a prime non-Γ factor.

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Conclusion

1. WIP: Most of those results have a generalization relative to a subalgebra B (time reversal, applications to free entropy relative to B and a completely positive map η).
2. Main Problem : Do we have $W^*(X, S_t) = W^*(X + S_t) * L(\mathbb{F}_\infty)$? (or something close to get absence of Cartan subalgebras result using [Ioana 2012] as in [Ioana-D 2012])
3. Especially, do we have $\delta_i \xi_j (X + S_t) \in M \otimes M^{op}$? or at least in all $L^p(M \otimes M^{op})$ (which is really likely equivalent to all higher derivatives in L^2) ?
4. Do we have for $T \geq t$:

$$||\xi_i(X + S_t)||_2^2 = ||\xi_i(X + S_T)||_2^2 + \int_t^T ||\delta_s(\xi_i(X + S_s))||_2^2 ds?$$

Thank you for your attention.

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Thank you for your attention.

Theorem (Variant of Peterson 2006, Th 3.3)

Let $Q \subset M$ be a von Neumann subalgebra with M finite. Assume that, for any projection $p \in Q' \cap M$, Qp is non-amenable, then $Q' \cap M \subset M$ is wCMW-rigid. More generally, for any derivation δ on M, if there is a wCMW-dilation, then there is another wCMW-dilation of δ converging uniformly on $(Q' \cap M^\omega)_1$.

Theorem (Variant of Peterson 2006, Th 3.5)

If $Q \subset M$ is a von Neumann subalgebra such that Q is diffuse and if the inclusion $Q \subset M$ is wCMW-rigid, then $W^*(N_M(Q)) \subset M$ is wCMW-rigid. More generally, any free ultrafilter ω, if $Q \subset M^\omega$ is a von Neumann subalgebra such that Q is diffuse, for any derivation δ if there is a wCMW-dilation α_t, β_t, such that $\alpha_t - \beta_t$ converges uniformly on Q, then there is another dilation of δ converging uniformly on $(W^*(N_{M^\omega}(Q) \cap M))_1$.