Quantum symmetric states on universal free product C*-algebras

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Classical Exchangeability

Definition

A sequence of (classical) random variables $x_1, x_2, \ldots$ is said to be *exchangeable* if

$$\mathbb{E}(x_{i(1)}x_{i(2)}\cdots x_{i(n)}) = \mathbb{E}(x_{\sigma(i(1))}x_{\sigma(i(2))}\cdots x_{\sigma(i(n))})$$

for every $n \in \mathbf{N}$, $i(1), \ldots, i(n) \in \mathbf{N}$ and every permutation σ of $\mathbf{N}$.

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for every $n \in \mathbf{N}$, $i(1), \ldots, i(n) \in \mathbf{N}$ and every permutation σ of $\mathbf{N}$.

That is, if the joint distribution of $x_1, x_2 \ldots$ is invariant under re-orderings.

De Finetti's Theorem

Theorem [de Finetti, 1937]

A sequence of random variables $x_1, x_2, \ldots$ is exchangeable if and only if the random variables are conditionally independent and identically distributed over its tail σ-algebra.

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Thus, the expectation $\mathbb{E}$ can be seen as an integral (w.r.t. a probability measure on the tail algebra) — that is, as a sort of convex combination — of expectations with respect to which the random variables $x_1, x_2, \ldots$ are independent and identically distributed (i.i.d.).

Symmetric states

Størmer extended this result to the realm of C*–algebras.

Definition

Consider the minimal tensor product $B = \bigotimes_1^\infty A$ of a C*–algebra A with itself infinitely many times. A state on B is said to be *symmetric* if it is invariant under the action of the group S_∞ by permutations of tensor factors.

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Theorem [Størmer, 1969]

The extreme points of $SS(A)$ are the infinite tensor product states, i.e. those of the form $\otimes_1^\infty \phi$ for $\phi \in S(A)$ a state of A. Moreover, $SS(A)$ is a Choquet simplex, so every symmetric state on B is an integral w.r.t. a *unique* probability measure of infinite tensor product states.

The quantum permutation group of Shuzhou Wang [1998]

The quantum permutation group $A_s(n)$

$A_s(n)$ is the universal unital C*–algebra generated by a family of projections $(u_{i,j})_{1\leq i,j\leq n}$ subject to the relations

$$\forall i \sum_j u_{i,j} = 1 \quad \text{and} \quad \forall j \sum_i u_{i,j} = 1. \tag{1}$$

It is a compact quantum group (with comultiplication, counit and antipode).

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Abelianization of $A_s(n)$

The universal unital C*–algebra generated by *commuting* projections $\tilde{u}_{i,,j}$ satisfying the analogous relations (1) is isomorphic to $C(S_n)$, the continuous functions of the permutation group S_n, with $\tilde{u}_{i,j}$ the characterisitc set of the permutations sending j to i. Thus, $C(S_n)$ is a quotient of $A_s(n)$ by a $*$-homomorphism sending $u_{i,j}$ to $\tilde{u}_{i,j}$.

Fully noncommutative version of permutation invariance

Invariance under quantum permutations

In a C*-noncommutative probability space (A, ϕ), the joint distribution of family $x_1, \ldots, x_n \in A$ is *invariant under quantum permtuations* if the natural coaction of $A_s(n)$ leaves the distribution unchanged. Concretely, this amounts to:

$$\phi(x_{i(1)} \cdots x_{i(k)})1 = \sum_{1 \le j(1),\ldots,j(k) \le n} u_{i(1),j(1)} \cdots u_{i(k),j(k)} \phi(x_{j(1)} \cdots x_{j(k)}) \in \mathbf{C}1 \subseteq A_s(n).$$

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Invariance under quantum permutations implies invariance under usual permuations

by taking the quotient from $A_s(n)$ onto $C(S_n)$.

Quantum exchangeable random variables and the tail algebra

Definition [Köstler, Speicher '09]

In a C*-noncommutative probability space, a sequence of random variables $(x_i)_{i=1}^{\infty}$ is *quantum exchangeable* if for every n, the joint distribution of $x_1, \ldots, x_n$ is invariant under quantum permutations.

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The *tail algebra* of the sequence is

$$\mathcal{T} = \bigcap_{N=1}^{\infty} W^*(\{x_N, x_{N+1}, \ldots\}).$$

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Proposition [Köstler '10] (existence of conditional expectation)

Let $(x_i)_{i=1}^{\infty}$ be a quantum exchangeable sequence in a W*-noncommutative probability space $(\mathcal{M}, \phi)$ where ϕ is faithful and suppose $\mathcal{M}$ is generated by the x_i. Then there is a unique faithful, ϕ–preserving conditional expectation E from $\mathcal{M}$ onto $\mathcal{T}$.

Quantum exchangeable ⇔ free with amalgamation over tail algebra

Theorem [Köstler, Speicher '09]

$(x_i)_{i=1}^{\infty}$ is a quantum exchangeable sequence if and only if the random variables are free with respect to the conditional expectation E (i.e., with amalgamation over the tail algebra).

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Theorem [D., Köstler]

Given any countably generated von Neumann algebra $\mathcal{A}$ and any faithful state ψ on $\mathcal{A}$, there is a W*–noncommutative probability space $(\mathcal{M}, \phi)$ with ϕ faithful and with a sequence $(x_i)_{i=1}^{\infty}$ of random variables that is quantum exchangeable with respect to ϕ, and so that their tail algebra $\mathcal{T}$ is a copy of $\mathcal{A}$ so that $\phi\restriction_{\mathcal{T}}$ is equal to ψ.

Change in perspective

Generalize in the direction of C*-algebras, like Størmer did

Instead of considering individual random variables, we consider a unital C^*–algebra A and a state ψ on the universal unital free product C*–algebra $\mathfrak{A} = *_1^\infty A$, with corresponding embeddings $\lambda_i : A \to \mathfrak{A}$, $(i \geq 1)$.

Definition

A state ψ on $\mathfrak{A}$ is *quantum symmetric* if the $*$–homomorphisms λ_i are quantum exchangeable with respect to ψ, in the sense that, for all $n \in \mathbf{N}$, $a_1, \ldots, a_k \in A$ and $1 \leq i(1), \ldots, i(k) \leq n$,

$$\psi(\lambda_{i(1)}(a_1) \cdots \lambda_{i(k)}(a_k))1$$
$$= \sum_{1 \leq j(1), \ldots, j(k) \leq n} u_{i(1),j(1)} \cdots u_{i(k),j(k)} \psi(\lambda_{j(1)}(a_1) \cdots \lambda_{j(k)}(a_k))$$
$$\in \mathbf{C}1 \subseteq A_s(n).$$

Quantum symmetric states yield freeness with amalgamation over the tail algebra

Notation

Let $\mathrm{QSS}(A)$ denote the set of quantum symmetric states on $\mathfrak{A} = *_1^\infty A$. It is a closed, convex subset of the set of all states on $\mathfrak{A}$.

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Proposition

Let $\psi \in \mathrm{QSS}(A)$. Let $\pi_\psi : \mathfrak{A} \to B(L^2(\mathfrak{A}, \psi))$ be the GNS representation, let $\mathcal{M}_\psi = W^*(\pi_\psi(\mathfrak{A}))$ and denote by $\hat{\psi}$ the GNS vector state $\langle \cdot \hat{1}, \hat{1} \rangle$ on $\mathcal{M}_\psi$. Consider the *tail algebra* $\mathcal{T}_\psi = \bigcap_{N=1}^\infty W^*(\bigcup_{i=N}^\infty \pi_\psi \circ \lambda_i(A))$. Then there is a $\hat{\psi}$-preserving conditional expectation E_ψ from $\mathcal{M}_\psi$ onto $\mathcal{T}_\psi$.

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Theorem

The subalgebras $\pi_\psi \circ \lambda_i(A)$ for $i \geq 1$ are free with respect to the conditional expectation E_ψ onto the tail algebra.

Conversely, freeness with amalgamation leads to quantum symmetric states

Recall $\mathfrak{A} = *_1^\infty A$.

Theorem

Let (B, ϕ) be a C*-noncommutative probability space and suppose $D \subseteq B$ is a unital C*–subalgebra with a conditional expectation $E : B \to D$ and let ρ be a state on B such that $\rho \circ E = \rho$. If $\pi : \mathfrak{A} \to B$ is a $*$–homomorphism such that the states $\rho \circ \pi \circ \lambda_i$ of A are the same for all i and the algebras $(\pi \circ \lambda_i)_{i=1}^\infty$ are free with respect to E, then $\psi = \rho \circ \pi \in \mathrm{QSS}(A)$.

Remarks

- We don't require faithfulness of ψ on $\mathfrak{A}$, nor of $\hat{\psi}$ on $\mathcal{M}_\psi$, nor of E_ψ on $\mathcal{M}_\psi$.
- Only classical exchangeability (not quantum exchangebility) is required for existence of a ψ-preserving conditional expectation $E_\psi : \mathcal{M}_\psi \to \mathcal{T}_\psi$ onto the tail algebra.
- Our proof are similar to those in [Köstler '10] and [Köstler, Speichter '09].
- Also Stephen Curran ['09] considered quantum exchangeability for sequences of $*$–homomorphisms of $*$-algebras and proved freeness with amalgamation; he did require faithfulness of a state, and used different ideas for his proofs.

Goals

To investigate $\mathrm{QSS}(A)$ as a compact, convex subset of $S(\mathfrak{A})$, to characterize its extreme points and to study certain convex subsets:

- the *tracial quantum symmetric states* $\mathrm{TQSS}(A) = \mathrm{QSS}(A) \cap TS(\mathfrak{A})$
- the *central quantum symmetric states* $\mathrm{ZQSS}(A) = \{\psi \in \mathrm{QSS}(A) \mid \mathcal{T}_\psi \subseteq Z(\mathcal{M}_\psi)\}$
- the *tracial central quantum symmetric states* $\mathrm{ZTQSS}(A) = \mathrm{ZQSS}(A) \cap \mathrm{TQSS}(A)$.

Description of $\mathrm{QSS}(A)$

There is a bijection $\mathcal{V}(A) \leftrightsquigarrow \mathrm{QSS}(A)$

where $\mathcal{V}(A)$ is the set of all quintuples $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ where

- $1_\mathcal{B} \in \mathcal{D} \subseteq \mathcal{B}$ is a von Neumann subalgebra and $E : \mathcal{B} \to \mathcal{D}$ is a normal conditional expectation
- $\sigma : A \to \mathcal{B}$ is a unital $*$–homomorphism
- ρ is a normal state on $\mathcal{D}$ so that the state $\rho \circ E$ of $\mathcal{B}$ has faithful GNS rep
- $\mathcal{B} = W^*(\sigma(A) \cup \mathcal{D})$
- $\mathcal{D}$ is the smallest unital von Neumann subalgebra of $\mathcal{B}$ such that $E(d_0\sigma(a_1)d_1 \cdots \sigma(a_n)d_n) \in \mathcal{D}$ for all $a_1, \ldots, a_n \in A$ and all $d_0, \ldots, d_n \in \mathcal{D}$.

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The bijection takes $(\mathcal{B}, \mathcal{D}, \mathrm{E}, \sigma, \rho) \in \mathcal{V}(A)$, constructs the W*–free product $(\mathcal{M}, F) = (*_{\mathcal{D}})_1^\infty(\mathcal{B}, E)$ with amalgamation over $\mathcal{D}$, and yields the quantum symmetric state $\rho \circ E \circ (*_1^\infty \sigma)$ on $\mathfrak{A} = *_1^\infty A$.

Description of $\mathrm{QSS}(A)$ (2)

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Under the bijection:

from $(\mathcal{B}, \mathcal{D}, \mathrm{E}, \sigma, \rho)$	$\mathcal{D}$	$\mathcal{M}$	$*_1^\infty \sigma$	F	$\rho \circ F$
from GNS rep of ψ	$\mathcal{T}_\psi$	$\mathcal{M}_\psi$	π_ψ	E_ψ	$\hat{\psi}$

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from GNS rep of ψ	$\mathcal{T}_\psi$	$\mathcal{M}_\psi$	π_ψ	E_ψ	$\hat{\psi}$

Technically, we need to let $\mathcal{V}(A)$ be the set of equivalence classes of quintuples, up to a natural notion of equivalence, and to avoid set theoretic difficulties we need to (and we can) restrict to $\mathcal{B}$ that are represented on some specific Hilbert space.

Extreme quantum symmetric states

Let $\partial_e(\mathrm{QSS}(A))$ be the set of extreme points of $\mathrm{QSS}(A)$.

Theorem

Let $\psi \in \mathrm{QSS}(A)$ correspond to $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$. Then $\psi \in \partial_e(\mathrm{QSS}(A))$ if and only if ρ is a pure state on $\mathcal{D}$.

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A very special form

A pure state ρ on a von Neumann algebra $\mathcal{D}$ is always of the form $\mathcal{D} = B(\mathcal{H}) \oplus \mathcal{N}$ and $\rho(a \oplus x) = \langle a\xi, \xi\rangle$ for a unit vector $\xi \in \mathcal{H}$.

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Examples of extreme quantum symmetric states

- free product states $\psi = *_1^\infty \phi$ for $\phi \in S(A)$; these correspond to $\mathcal{D} = \mathbf{C}$.
- we construct an example $\psi \in \partial_e(\mathrm{QSS}(\mathbf{C} \oplus \mathbf{C}))$ with $\mathcal{D} = \mathbf{C} \oplus L^\infty([0,1])$.

Extreme tracial quantum symmetric states

Let $\mathrm{TQSS}(A)$ be the set of all $\psi \in \mathrm{QSS}(A)$ that are traces on $\mathfrak{A} = *_1^\infty A$ and let $\partial_e(\mathrm{TQSS}(A))$ be the set of extreme points of $\mathrm{TQSS}(A)$.

Theorem

Let $\psi \in \mathrm{TQSS}(A)$ correspond to $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$. Let $R(E) = \{\tau \in TS(\mathcal{D}) \mid \tau \circ E \in TS(\mathcal{B})\}$. Then $\psi \in \partial_e(\mathrm{TQSS}(A))$ if and only if ρ is an extreme point of $R(E)$.

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Corollary

If either $\mathcal{D}$ or $\mathcal{B}$ is a factor, then $\psi \in \partial_e(\mathrm{TQSS}(A))$.

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- we construct an example $\psi \in \partial_e(\mathrm{TQSS}(\mathbf{C} \oplus \mathbf{C}))$ with $\mathcal{D} = M_2(\mathbf{C})$, so tail algebras can be noncommutative also when $A = \mathbf{C} \oplus \mathbf{C}$.
- we construct an example $\psi \in \partial_e(\mathrm{TQSS}(\mathbf{C} \oplus \mathbf{C}))$ with $\mathcal{D} = \mathbf{C} \oplus \mathbf{C}$ and $\mathcal{B} = M_2(\mathbf{C}) \oplus M_2(\mathbf{C})$, so extreme tracial quantum symmetric states can occur when neither $\mathcal{B}$ nor $\mathcal{D}$ is a factor.

Central quantum symmetric states

$\mathrm{ZQSS}(A) =$ the set of all $\psi \in \mathrm{QSS}(A)$ whose tail algebra $\mathcal{T}_\psi$ lies in the center of $\mathcal{M}_\psi$.

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Theorem

Both $\mathrm{ZQSS}(A)$ and $\mathrm{ZTQSS}(A)$ are compact, convex subsets of $\mathrm{QSS}(A)$ and both are Choquet simplices. Their extreme points are, respectively, the free product states and the free product traces:

$$\partial_e(\mathrm{ZQSS}(A)) = \{*_1^\infty \phi \mid \phi \in S(A)\},$$
$$\partial_e(\mathrm{ZTQSS}(A)) = \{*_1^\infty \tau \mid \tau \in TS(A)\}.$$

The previous result is in the spirit of Størmer's result; it says that each central quantum symmetric state ψ can be written as an integral

$$\psi = \int_{S(A)} (*_1^\infty \phi)\, d\mu(\phi)$$

of free product states for a *unique* Borel probability measure μ on $S(A)$, and in the case that ψ is a trace, $\mathrm{supp}(\mu) \subseteq TS(A)$.

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Open problem

Is $\mathrm{TQSS}(A)$ a Choquet simplex?

Proof that $\mathrm{ZQSS}(A)$ is closed and that $\partial_e(\mathrm{ZQSS}(A)) = \{*_1^\infty \phi \mid \phi \in S(A)\}$.

Step 1

Note that $\phi \mapsto *_1^\infty \phi$ is a homeomorphism from $S(A)$ into $\partial_e(\mathrm{QSS}(A))$.

Step 2

Show $\mathrm{ZQSS}(A) \subseteq \overline{\mathrm{conv}}\{*_1^\infty \phi \mid \phi \in S(A)\}$.

If $\psi \in \mathrm{ZQSS}(A)$ comes from $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$, then $\mathcal{D}$ lies in the center of $\mathcal{B}$, so ρ is a state on $\mathcal{D} \cong C(X)$ and is approximately a convex combination of point masses. Using a result from [D., Köstler], each $(\text{point mass}) \circ E \circ *_1^\infty \sigma : \mathfrak{A} \to \mathbf{C}$ is a free product state of the form $*_1^\infty \phi$.

Proof (continued).

Step 3

Show $\mathrm{ZQSS}(A) \supseteq \overline{\mathrm{conv}}\{*_1^\infty \phi \mid \phi \in S(A)\}$.

It is easy to see $\mathrm{ZQSS}(A) \supseteq \mathrm{conv}\{*_1^\infty \phi \mid \phi \in S(A)\}$. But even if $\psi_i \in \mathrm{QSS}(A)$ and $\psi_i \to \psi$ and we understand the tail algebras of each ψ_i, how do we understand the tail algebra of ψ?

Proof (continued).

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We don't answer this general question. Instead, since $\{*_1^\infty \phi \mid \phi \in S(A)\}$ is compact, for every $\psi \in \overline{\mathrm{conv}}\{*_1^\infty \phi \mid \phi \in S(A)\}$ there is a *Borel* probability measure μ on $S(A)$ such that $\psi(x) = \int_{S(A)} (*_1^\infty \phi)(x)\, d\mu(\phi)$.

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Proof (continued).

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These three steps show $\mathrm{ZQSS}(A)$ is compact, convex and $\partial_e(\mathrm{ZQSS}(A)) = \{*_1^\infty \phi \mid \phi \in S(A)\}$.

Proof (continued).

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Proof that $\mathrm{ZQSS}(A)$ is a Choquet simplex.

As remarked earlier, since $\partial_e(\mathrm{ZQSS}(A)) = \{*_1^\infty \phi \mid \phi \in S(A)\}$ is compact, for every $\psi \in \mathrm{ZQSS}(A)$ there is a Borel probability measure μ on $S(A)$ so that

$$\psi(x) = \int_{S(A)} (*_1^\infty \phi)(x)\, d\mu(\phi)$$

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If $\lambda_i : A \to *_1^\infty A$ is the embedding to the i-th copy, then $(*_1^\infty \phi)(\lambda_1(a_1) \cdots \lambda_k(a_k)) = \prod_1^k \phi(a_j)$, so

$$\int_{S(A)} \prod_1^k \phi(a_j)\, d\mu(\phi) = \psi(\lambda_1(a_1) \cdots \lambda_k(a_k)) = \int_{S(A)} \prod_1^k \phi(a_j)\, d\nu(\phi).$$

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Thus, the linear functionals $\int \cdot\, d\mu$ and $\int \cdot\, d\mu$ agree on the closed subalgebra of $C(S(A))$ generated by the evaluations $\phi \mapsto \phi(a)$, $(a \in A)$. By Stone–Weierstrass $\mu = \nu$. QED

Proof that $\mathrm{ZTQSS}(A)$ is a Choquet simplex and $\partial_e(\mathrm{ZTQSS}(A)) = \{*_1^\infty \tau \mid \tau \in TS(A)\}$.

Recall $\mathrm{ZTQSS}(A) = \mathrm{ZQSS}(A) \cap TS(\mathfrak{A})$. Suppose $\psi \in \mathrm{ZTQSS}(A)$ and μ is the (unique) Borel measure on $S(A)$ such that $\psi = \int_{S(A)} (*_1^\infty \phi)\, d\mu(\phi)$. It will suffice to show $\mathrm{supp}(\mu) \subseteq TS(A)$.

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Let $a \in A$ $\|a\| \leq 1$ and let ω denote the push–forward measure of μ under the map $S(A) \to [0,1]^2$ given by $\phi \mapsto (\phi(a^*a), \phi(aa^*))$. It will suffice to show that the support of ω lies in the diagonal.

Proof (continued).

Recall $|a| = (a^*a)^{1/2}$ and $|a^*| = (aa^*)^{1/2}$. Let $x = \lambda_1(|a|)\lambda_2(a)$ and $y = \lambda_1(|a^*|)\lambda_2(a^*)$. Then for all $\phi \in S(A)$,

$$(*_1^\infty \phi)(x^*x) = \phi(a^*a)^2, \qquad (*_1^\infty \phi)(xx^*) = \phi(a^*a)\phi(aa^*),$$
$$(*_1^\infty \phi)(y^*y) = \phi(aa^*)^2, \qquad (*_1^\infty \phi)(yy^*) = \phi(a^*a)\phi(aa^*).$$

Proof (continued).

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Thus, we have

$$\int_{[0,1]^2} s^2 \, d\omega(s,t) = \psi(x^*x) = \psi(xx^*) = \int_{[0,1]^2} st \, d\omega(s,t),$$
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From these identities, we get $\int (s-t)^2 \, d\omega(s,t) = 0$ and we conclude that the support of ω lies in the diagonal of $[0,1]^2$.

Proof (continued).

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