# Asymptotic Approximation and Inverse Problems for Sums of Free Variables

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Fields Institute, July 23, 2013

# Topics

- Approximations of Distributions and Densities in the Free CLT
- Expansions of Free and Classical Entropy for Sums
- (Inverse Characterization Problems for Sums of Free Variables)

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## Additive Free Convolution: Analytic Approach

$\mu \in \mathcal{M}$: distribution on $\mathbb{R}$, on $\mathbb{C}_+$: (upper halfplane)

$$F_\mu(z) := \Big(\int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}\Big)^{-1},$$

is a Nevanlinna function: that is ( $\mathbb{C}_+ \to \mathbb{C}_+$ ),

such that $\lim_z F_\mu(z)/z = 1$, $\Re(z)/\Im(z)$ bounded, defines class: $\mathcal{F}$.

Define:

**(1)** $\phi_\mu(z) := F_\mu^{(-1)}(z) - z$, (characterizing $\mu$ ),

(Bercovici, Pata, Biane, Voiculescu 96,99), $F_\mu^{(-1)}(z)$ defined on a $\mu$-dependent subset

**(2)** $\phi_{\nu_1 \boxplus \nu_2} = \phi_{\nu_1} + \phi_{\nu_2}$,

on the **common** domain of definition of $\phi_{\nu_1}, \phi_{\nu_2}$

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## Expansions of C-Transforms in the Free CLT I

$$G_\mu(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{z-x}$$

$\mu_n((-\infty, x]) := \mu^{n\boxplus}((-\infty, x\sqrt{n}]), \quad m_1(\mu) = 0, \quad m_2(\mu) = 1,$

formal expansion using free cumulants

$\alpha_3(\mu) = m_3(\mu), \quad \alpha_4(\mu) = m_4(\mu) - 2, \ldots$

$$G_{\mu_n}(z) = G_w(z) + \sum_{k=1}^{\infty} \frac{B_k(G_w(z))}{n^{k/2}}, \quad \text{where}$$

$$B_k(z) := \sum c_{p,m} \frac{z^p}{(1/z - z)^m}, \quad c_{p,m} = c_{p,m}(\alpha_3(\mu), \ldots, \alpha_{k+2}(\mu)).$$

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$$B_1(z) = \alpha_3(\mu)\frac{z^3}{z^{-1}-z}, \; U_m(x) = U_m(\cos\theta) := \frac{\sin(m+1)\theta}{\sin\theta}, \; m = 1,2,\ldots$$

$$B_1(G_w(z)) = \frac{\alpha_3(\mu)}{\sqrt{z^2-4}} G_w^3(z) = \alpha_3(\mu)\int_{-2}^{2}\frac{1}{z-x}\, d\Big(\frac{1}{3}U_2(\frac{x}{2})p_w(x)\Big), \quad z \in \mathbb{C}^+$$

$$p_w(x) := \frac{1}{2\pi}\sqrt{(4-x^2)_+}$$

$$B_2(z) = (\alpha_4 - \alpha_3^2)\frac{z^4}{1/z - z} + \alpha_3^2\Big(\frac{z^5}{(1/z-z)^2} + \frac{z^2}{(1/z-z)^3}\Big).$$

If $\alpha_3(\mu) \neq 0$: $B_2(G_w)$ is not a Cauchy-Transform of a finite signed measure.

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$$\text{a.c.} - \mu_{\mathrm{a,b,d}}(\mathrm{x}) \quad = \quad \frac{\sqrt{4(1-d)-(1-b)^2(x-a)^2}}{2\pi(bx^2+a(1-b)x+1-d)},$$

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*Wigner half-circle:* (= free Gaussian) : $\mu_{0,0,0} = w$

*Marchenko-Pastur:* (= free Poisson) : $\mu_{a,0,0}$, $a \neq 0$

*Free Pascal:* (= Neg. Binomial) : $\mu_{a,b,0}$, $a \neq 0, b > 0$, $\Delta > 0$

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Discriminant $\Delta := a^2(1-b)^2 - 4b(1-d)$.

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Th.: [Chistyakov, G. PTRF]

free add. convolution of ident. $\mu$

s.th. $\beta_q = \beta_q(\mu) < \infty, \quad q \geq 5$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$.

There $\exists\, c > 0$ s.th. $\forall \quad n \geq m_4$

$$\sup_{x \in \mathbb{R}} |F_n(x) - \mu_{a_n, b_n, d_n}((-\infty, x]) - \frac{1}{n}\nu((-\infty, x])| \leq c\beta_5 n^{-3/2}.$$

$$\leq \eta_n n^{-(q-2)/2},$$

where $|\eta_n| \leq 1$ and if $4 < q < 5$, $\lim_n \eta_n = 0$.

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$$\sup_{x \in \mathbb{R}} |F_n(x) - \mu_{a_n,b_n,d_n}((-\infty,x]) - \frac{1}{n}\nu((-\infty,x])| \leq c\beta_5 n^{-3/2}.$$
$$\leq \eta_n n^{-(q-2)/2},$$

where $|\eta_n| \leq 1$ and if $4 < q < 5$, $\lim_n \eta_n = 0$.

$\nu$ signed measure with density:

$$p_\nu(x) := \frac{1}{2\pi}(x^2-1)\sqrt{(4-x^2)_+}$$
$$a_n := \frac{m_3}{\sqrt{n}}, \quad b_n := d_n - \frac{1}{n}, \quad d_n := \frac{m_4 - m_3^2}{n}$$

Th.: [Chistyakov, G. PTRF]

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# Approximations in the Free CLT II

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## Expansion of Densities

If $\mathrm{supp}(\mu) \subset [-\mathrm{L}, \mathrm{L}]$ , $\mathrm{supp}(\mu_\mathrm{n}) \subset [-2 - \frac{\mathrm{L}}{\sqrt{\mathrm{n}}}, 2 + \frac{\mathrm{L}}{\sqrt{\mathrm{n}}}]$ for $n \geq 1$, (Voiculescu(1986),

For $n \geq n_0$ and $\mu$ nontrivial, $\mu_n'(x) = p_n(x)$ exists. (Belinschi-Bercovici (2004)).

Recall $d_n := \frac{m_4 - m_3^2}{n}, b_n = d_n - \frac{1}{n}, a_n = m_3 n^{-1/2}$, $E_n := \frac{1-b_n}{\sqrt{1-d_n}} = 1 - O(n^{-1})$.

**Theorem** (Chistyakov-G. PTRF )

Assume $\mathrm{supp}(\mu) \subset [-\mathrm{L}, \mathrm{L}]$, $m_1(\mu) = 0$, $m_2(\mu) = 1$.

Then for $n \geq n_0$, $|\theta| \leq 1$, and $h := \tilde{c}(\mu) n^{-3/2}$ ,

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r.v. $X$ with density $p$, $\mathbf{E}X^2 < \infty$, $a := \mathbf{E}\,X$, $\sigma^2 := Var(X)$,

entropy of $X$: $h(X) := -\int p(x) \log p(x)\, dx$, $\mathcal{L}(Z) = \Phi_{a,\sigma}$

Entropy distance to normality:

$$\begin{aligned} D(X) &= h(Z) - h(X) = \int_{-\infty}^{+\infty} p(x) \log \frac{p(x)}{\varphi_{a,\sigma}(x)}\, dx \in [0, +\infty], \\ D(a + bX) &= D(X), \quad a \in \mathbb{R}, \quad b \neq 0, \quad \text{(translation and scale)} \\ D(X) &= 0, \quad \text{iff} \quad X \quad \text{is normal.} \end{aligned}$$

$D(X)$ shortest Kullback-Leibler distance of $\mathcal{L}(X)$ to **any** normal law.

$X, X_1, \dots, X_n$ i.i.d. $\mathbf{E}X^2 < \infty$, $S_n := X_1 + \dots + X_n$.

$n \mapsto D(S_n)$ is monotone: Artstein,Ball,Barron,Madiman (02, 07).

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## Entropic CLT Expansions

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$$\chi(\nu) := \int\int_{\mathbb{R}\times\mathbb{R}} \log|x-y|\,\nu(dx)\nu(dy) + \chi_0, \quad \chi_0 = \frac{3}{4} + \frac{1}{2}\log 2\pi,$$

maximized by Wigner's $w$, for all $m_1(\nu) = 0$, $m_2(\nu) = 1$,

Entropic Distance for such $\nu$:

$$D(\nu) := \chi(w) - \chi(\nu)$$

$\mu_n := \mathcal{D}((X_1 + \dots X_n)/\sqrt{n})$, $X_1, \dots, X_n$, identically distr. **free** variables,

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$D(\mu_n) \to 0$, Wang (2010), monotonicity: Shlyakhtenko (2007)

**Th.** (Chistyakov-G. (2012/13), arxiv)

For $n \geq n_1$, $\mathrm{E}|X_1|^s < \infty$

$$D(\mu_n) = \frac{m_3^2}{6} n^{-1} + \frac{\theta_n}{n}, \quad \text{where}$$

$\theta_n = O(n^{-1/2}),\ s = \infty,\ \theta_n = O(n^{-1/5}),\quad s = 8,\quad \theta_n = o(1),\ s > 5$

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# Free Entropy Distance

$$\chi(\nu) := \int\int_{\mathbb{R}\times\mathbb{R}} \log|x-y|\,\nu(dx)\nu(dy) + \chi_0, \quad \chi_0 = \frac{3}{4} + \frac{1}{2}\log 2\pi,$$

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## Asymptotic Expansions in the CLT in Fisher-Information Distance

$X$ with density $p$: $$I(X) := I(p) := \int_{-\infty}^{+\infty} \frac{p'(x)^2}{p(x)}\, dx,$$

or $I(X) := +\infty$. Let $\mathrm{E}\, X = a$, $\mathrm{Var}(X) = \sigma^2$ Cramer-Rao:

$$I(X) \geq I(Z), \quad \text{where } \mathcal{D}(Z) = N(a, \sigma^2) \text{ with } = \text{ iff } \mathcal{D}(X) = \mathcal{D}(Z).$$

$I(S_n)$ is monotone in $n$. Relative Fisher information

$$I(X||Z) := I(X) - I(Z) = \int_{-\infty}^{+\infty} \left( \frac{p'(x)}{p(x)} - \frac{\varphi'_{a,\sigma}(x)}{\varphi_{a,\sigma}(x)} \right)^2 p(x)\, dx$$

dominates the relative entropy (cf. Stam (1959)).

$X_j$ i.i.d. as above. Linnik (1959) Barron and Johnson (2004) proved that

$$I(S_n) \to I(Z), \quad \text{as } n \to \infty,$$

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**Th.** (Bobkov-Chistyakov-G.(2012), PTRF)

Let $\beta_s := \mathrm{E}\ |X_1|^s < +\infty$, $s \geq 2$, $s \in \mathbb{N}$

Assume $I(S_{n_0}) < +\infty$, for some $n_0$. Then, with $s_0 = [(s-2)/2]$,

$$I(S_n||Z) = \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots + \frac{c_{s_0}}{n^{s_0}} + o\left(n^{-\frac{s-2}{2}} (\log n)^{-\frac{(s-3)_+}{2}}\right),$$

as $n \to \infty$, with coefficients $c_j$ like $c_1 = \frac{1}{2}\gamma_3^2$.

$$I(S_n||Z) \leq \frac{c(\beta_4, I(X_1))}{n}, \quad s = 4.$$

For $s = 6$, the result involves a coefficient $c_2 = c_2(\gamma_3, \gamma_4, \gamma_5)$.

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Let $\mu(dx) = p(x)\,dx$,

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Let $\mu(dx) = p(x)\, dx$,

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$\mu_n := \mathcal{D}((X_1 + \ldots + X_n)/\sqrt{n})$ , $\quad X_1, \ldots, X_n,$

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**Th.** (Chistyakov-G. (2012/13), arxiv)

For $n \geq n_1$,

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## Proof: Subordinating Fct.

[Voiculescu (1992), Biane (1998), Chistyakov & G. (2005), Belinschi (2006)]

Given $\nu_1, \nu_2$: exist unique functions $f_j(z) \in \mathcal{F}$, s. th.

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## Higher order equations and Meixner Families

If $m_3(\mu) = 0$ then $d_n = m_4 n^{-1}, b_n = d_n - n^{-1}$

$$p_{\mu_{n2}}(x) := \frac{1}{2\pi}\frac{\sqrt{(4(1-d_n)-(1-b_n)^2x^2)_+}}{b_n x^2 + 1 - d_n}, \quad x \in \mathbb{R}$$

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**Theorem** ( Darmois, Skitovich):

$L_\alpha \perp L_\beta$ (independent) $\Rightarrow X_j$ Gaussian, if $\alpha_j \beta_j \neq 0$.

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$L_\alpha \perp L_\beta$ (independent) $\Rightarrow X_j$ Gaussian, if $\alpha_j \beta_j \neq 0$.

Nica (96) Distributional invariance under rotations implies Wigner

Lehner (04): $n = 3$ : Darmois-Skitovich fails

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$\alpha_j,\ \beta_j \in \mathbb{R}$ with $\alpha_j\beta_j \neq 0$ and $\frac{\beta_j}{\alpha_j} \neq \frac{\beta_s}{\alpha_s}$ for $j, s = 1, \dots, m$, where $m \leq n$, and $\alpha_j\beta_j = 0$ for $j = m+1, \dots, n$. The linear statistics

$$L_\alpha = \alpha_1 T_1 + \cdots + \alpha_n T_n, \quad L_\beta = \beta_1 T_1 + \cdots + \beta_n T_n$$

are free if and only if the distributions $\mu_{T_1}, \dots, \mu_{T_m}$ have compact supports and the free cumulants $\kappa_s(T_j)$, $j = 1, \dots, m$, satisfy the relations:

$s \in \{2, \dots, m\}$ : For all $(l, t) \in \mathbb{N}^2$ such that $l + t = s$ it holds:

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## Measures with Finitely Many Nonzero Free Cumulants

For $\kappa_1, \ldots, \kappa_m$ introduce $\quad \varphi(z) := \kappa_1 + \frac{\kappa_2}{z} + \cdots + \frac{\kappa_m}{z^{m-1}}, \quad z \in \mathbb{C} \setminus \{0\},$

and $\Omega_\varphi$ as component of $\quad \{z \in \mathbb{C}^+ : \Im(z + \varphi(z)) > 0\} \quad$ containing $\infty$.

**Theorem:** (Chistyakov, G., 2011, arxiv) $\{\kappa_n\}_{n=1}^{\infty} \in \mathbb{R}$ with

$\kappa_n = 0, \; n \geq m+1, \; m \geq 2,$

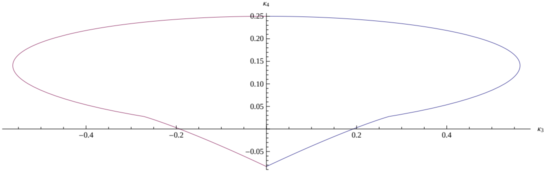
are free cumulants of some p-measure with compact support

if and only if every Jordan curve, contained in $\mathbb{C}^+ \cup \mathbb{R}$, connecting 0 and $\infty$,

contains a point of the boundary of $\Omega_\varphi$.

Consequences:

- regions in the $(\kappa_3, \kappa_4)$-plane for cumulants $0, 1, \kappa_3, \kappa_4, 0, \ldots$



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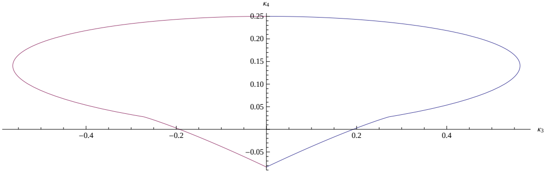

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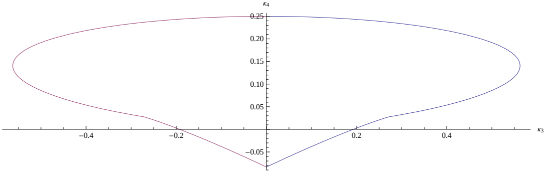

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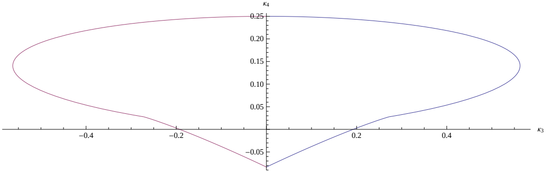

## Characterization of Semicircular Laws

For $z \in \mathbb{C}$ and $\alpha_j, \beta_j \in [-1, 1]$, $j = 1, \dots, n$, consider:

$$\Lambda_1(z) = \sum_{k=1}^{n} |\alpha_k|^z - \sum_{k=1}^{n} |\beta_k|^z \quad \text{and} \quad \Lambda_2(z) = \sum_{k=1}^{n} \alpha_k^z - \sum_{k=1}^{n} \beta_k^z.$$

**Theorem:** (Chistyakov, G., 2011, arxiv)

Let $T_1, \dots, T_n$ be free and identically distributed with $T_j \overset{D}{\sim} \mu$.

Assume $\Lambda_1(z) \not\equiv 0$. The statements:

(1) $\mu$ is a semicircular measure

(2) $L_\alpha \overset{D}{=} L_\beta$

are equivalent, if and only if the following conditions are satisfied:

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▸ classical

## Classical Conditions (Linnik, Zinger)

**Theorem:** $X_1, \ldots, X_n$ be i.i.d. with $X_j \stackrel{D}{\sim} \mu$. The statements:

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## Classical Conditions (Linnik, Zinger)

Theorem: $X_1, \dots, X_n$ be i.i.d. with $X_j \overset{D}{\sim} \mu$.The statements:

(1) $\mu$ is a normal law

(2) $L_\alpha \overset{D}{=} L_\beta$

are equivalent, if and only if:

(a) $\Lambda_1(2) = 0$,

(b) all positive zeros $s$ of $\Lambda_1$ , with $s = 4k$, $k \in \mathbb{N}$, are simple zeros,

(c) all positive zeros $s$ of $\Lambda_1$, with $s = 4k + 2$, $k \in \mathbb{N} \cup \{0\}$, have multiplicity at most two; a zero of multiplicity two is unique and the largest positive zero,

(d) if $\Lambda_1$ has a positive zero $s$, which is not an even integer, then it is unique, simple, and the largest positive zero, and $[s/2]$ is odd.

free

Thank you!