Hardy classes on non-commutative unit balls

Joint work with Victor Vinnikov

Non-commutative functions

- J. L. Taylor (Adv. in Math. ’72)
- D-V. Voiculescu (Asterisque ’95, also Jpn. J. of Math., ’08, Crelles ’09) (∼fully matricial functions)
- V. Vinnikov, D.S. Kaliuzhnyi-Verbovetskyi, M. P., S. Belinschi (2009 -’13)
- M. Aguiar (2011), M. Anshelevich (2011), B.Solel, P. Muhly (2013)

$\mathcal{V}$= vector space over $\mathbb{C}$;

- the *non-commutative space over* $\mathcal{V}$: $\mathcal{V}_{\text{nc}} = \coprod_{n=1}^{\infty} \mathcal{V}^{n \times n}$
- *noncommutative sets*: $\Omega \subseteq \mathcal{V}_{\text{nc}}$ such that $X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$ for all $X \in \Omega_n$, $Y \in \Omega_m$, where $\Omega_n = \Omega \cap \mathcal{V}^{n \times n}$
- *upper admissible sets*: $\Omega \subseteq \mathcal{V}_{\text{nc}}$ such that for all $X \in \Omega_n$, $Y \in \Omega_m$ and all $Z \in \mathcal{V}^{n \times m}$, there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, with

$$\begin{bmatrix} X & \lambda Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Examples of upper-admissible sets:

- $\Omega = \mathrm{Nilp}\,\mathcal{V}$ = the set of nilpotent matrices over $\mathcal{V}$
- If $\mathcal{V}$ is a Banach space and Ω is a non-commutative set, open in the sense that $\Omega_n \subseteq \mathcal{V}^{n\times n}$ is open for all n,
  then Ω is upper admissible.
- Noncommutative upper/lower half-planes over a C*-algebra $\mathcal{A}$:

$$\mathbb{H}^+(\mathcal{A}_{\text{nc}}) = \{a \in \mathcal{A}_{\text{nc}} : \Im a > 0\}$$
$$\mathbb{H}^-(\mathcal{A}_{\text{nc}}) = \{a \in \mathcal{A}_{\text{nc}} : \Im a < 0\}$$

$\Omega \subseteq \mathcal{V}_{\text{nc}}$ = non-commutative (upper admissible) set

Noncommutative function:
$f\colon \Omega \to \mathcal{W}_{\text{nc}}$ such that

- $f(\Omega_n) \subseteq M_n(\mathcal{W})$
- f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$ for all $X \in \Omega_n$, $Y \in \Omega_m$.
- f respects similarities: $f(TXT^{-1}) = Tf(X)T^{-1}$ for all $X \in \Omega_n$ and $T \in \mathrm{GL}_n(\mathbb{C})$ such that $TXT^{-1} \in \Omega_n$.

Equivalently, f respects *intertwinings*:
if $X \in \Omega_n$, $Y \in \Omega_m$, $S \in \mathbb{C}^{n\times m}$ such that $XS = SY$, then

$$f(X)S = Sf(Y)$$

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Examples of nc-functions:

- *non-commutative polynomials*

  $\mathcal{V} = \mathcal{A}^m, \mathcal{W} = \mathcal{A}$

  $f(X_1, \ldots, X_m) = X_1X_3 - X_3X_1 + b_1X_2X_4b_2X_5$

  N.B.: *A nc polynomial is determined uniquely by this type of nc function*

Examples of nc-functions:

- *free holomorphic functions (G. Popescu)*

$$f(X_1, \ldots, X_n) = \sum_{m=1}^{\infty} \sum_{\mathbf{i}=(i_1,\ldots,i_m)} A_{\mathbf{i}} X_{i_1} \cdots X_{i_m}$$

where $X_{i_1}, \ldots, X_{i_n}$ are free elements in some operator algebra

- the generalized moment series of $X \in \mathcal{A}$

$\phi : \mathcal{A} \longrightarrow \mathcal{D}$ cp $\mathcal{B}$-bimodule map

$\widetilde{\phi}((\mathbb{1} - X\cdot)^{-1}) = M_X(\cdot) = (M_{n,X})_n$, where $\widetilde{\phi} = (1_n \otimes \phi)_n$ is the fully matricial extension of ϕ.

$$M_{n,X}(b) = \sum_{k=0}^{\infty} (1_n \otimes \phi)([\mathcal{X} \cdot b]^k),$$

Difference-differential calculus

Nc functions admit a nice differential calculus. The difference-differential operators can be calculated directly by evaluation on block-triangular matrices.

$$f(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}) = \begin{bmatrix} f(X) & \Delta_R f(X,Y)(Z) \\ 0 & f(Y) \end{bmatrix}$$

The operator $Z \mapsto \Delta_R f(X,Y)(Z)$ is linear and

$$f(Y) = f(X) + \Delta_R(X,Y)(X-Y)$$

The Taylor-Taylor expansion:
If $f : \Omega \longrightarrow \mathcal{W}_{\text{nc}}$ is a non-commutative function, Ω=upper-admissible set, $X \in \Omega_n$. Then for each N and $X \in \Omega_n$ we have that

$$f(Y) = \sum_{k=0}^{N} \Delta_R^k f(\underbrace{X, \ldots, X}_{k+1 \text{ times}})(\underbrace{X - Y, \ldots, X - Y}_{k \text{ times}})$$
$$+\Delta_R^{N+1} f(\underbrace{X, \ldots, X}_{N+1 \text{ times}}, Y)(\underbrace{X - Y, \ldots, X - Y}_{N+1 \text{ times}})$$

$$f\left(\begin{bmatrix} X & Z_1 & 0 & \cdots & 0 \\ 0 & X & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & X & Z_k \\ 0 & \cdots & \cdots & 0 & Y \end{bmatrix}\right)$$

$$= \begin{bmatrix} f(X) & \Delta_R f(X,X)(Z_1) & \cdots & \cdots & \Delta_R^k f(X,\dots,X,Y)(Z_1,\dots,Z_k) \\ 0 & f(X) & \ddots & & \Delta_R^{k-1} f(X,\dots,X,Y)(Z_2,\dots,Z_k) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & f(X) & \Delta_R f(X,Y)(Z_k) \\ 0 & \cdots & \cdots & 0 & f(Y) \end{bmatrix}.$$

Moreover, if $0, X \in \Omega$ and

- $\mathcal{V}$ is finite dim,
- $\mathcal{W}$ is a Banach space,
- f is a nc-function locally bdd on slices separately in every matrix dimension

then

$$f(X) = \sum_{k=0}^{\infty} \widetilde{\Delta_R^k f}(\underbrace{0, \ldots, 0}_{k+1})(\underbrace{X, \ldots, X}_{k})$$

where $\widetilde{\Delta_R^k f}(\underbrace{0, \ldots, 0}_{k+1})$ are the fully matricial extension of the multilinear maps $\Delta_R^k f(\underbrace{0, \ldots, 0}_{k+1}) \colon \mathcal{V}^k \longrightarrow \mathcal{W}$ and series converges absolutely and uniformely (in fact, normally) on compacta of a completely circular set around $0 \cdot I_n$ contained in Ω_n, for all n.

Examples of nc-functions:

- *the generalized Cauchy transform* of X: $\mathcal{G}_X$

  $\phi : \mathcal{A} \longrightarrow \mathcal{D}$ cp $\mathcal{B}$-bimodule map

  $\mathcal{G}_X = (G_X^{(n)})_n$, where

  $$G_X^{(n)} : \mathbb{H}^+(M_n(\mathcal{B})) \ni b \mapsto G_X^{(n)}(b) = \phi \otimes 1_n[(b - X \otimes 1_n)^{-1}] \in \mathbb{H}^-(M_n(\mathcal{D}))$$

- *the non-commutative R-transform of* X: $\mathcal{R}_X$

$$M_X(b) - \mathbb{1} = R_\nu(bM_\nu(b))$$

Applications in Free Probability Theory

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Framework: finite dimensional vector spaces: $(\mathbb{C}^m)_{\text{nc}}$

Operator space structure on $\mathbb{C}^m$: - A collection of norms $\|\cdot\| = \{\|\cdot\|_n \text{on } (\mathbb{C}^m)^{n\times n}\}$ such that

- $\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\}$
- $\|TXS\|_m \leq \|T\|\|X\|_n\|S\|$

$X \in \mathbb{C}^{n\times n}, Y \in \mathbb{C}^{m\times m}, T \in \mathbb{C}^{m\times n}, S \in \mathbb{C}^{n\times m}$.

We shall be concerned with the following two operator space structures on $\mathbb{C}^m$:

- $\|X\|_\infty = \max\{\|X_1\|, \dots, \|X_m\|\}$
- $\|X\|_2 = \|\sum_{i=1}^m X_i^* X_i\|^{\frac{1}{2}}$

for $X = (X_1, X_2, \dots, X_m) \in (\mathbb{C}^{n\times n})^m$

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- For the norm $\|\cdot\|_\infty$, the non-commutative unit ball is

$$(\mathbb{D}^m)_{\mathrm{nc}} = \coprod_{n=1}^{\infty} \{(X_1, \ldots, X_m) \in (\mathbb{C}^{n\times n})^m : \|X_j\| < 1\}$$

with distinguished boundary

$$\mathrm{bd}(\mathbb{D}^m)_{\mathrm{nc}} = \coprod_{n=1}^{\infty} \{(X_1, \ldots, X_m) \in (\mathbb{C}^{n\times n})^m : \ X_j^* X_j = I_n\} = \mathcal{U}(n)^m$$

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On $(\mathrm{bd}(\mathbb{D}^m)_{\mathrm{nc}})_n = \mathcal{U}(n)^m$ there is the canonical Haar product measure.

On $(\mathrm{bd}(\mathbb{B}^m)_{\mathrm{nc}})_n \simeq \mathcal{U}(mn)/\mathcal{U}((m-1)n)$ there exists also a canonical $\mathcal{U}(mn)$-invariant Radon measure ν_n of mass 1.

For $f \in \mathrm{Alg}\{u_{i,j}, \overline{u_{i,j}} : 1 \leq i \leq n, 1 \leq j \leq mn\}$, the measure ν_n is actually easy to describe:

$$\int_{(\mathrm{bd}(\mathbb{B}^m)_{\mathrm{nc}})_n} f(X) d\nu_n(X) = \int_{\mathcal{U}(mn)} f(U) d\mathcal{U}_{mn}(U)$$

for $d\mathcal{U}_N$ the Haar measure on $\mathcal{U}_N$.

The Hardy H^2 spaces:

$$H^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}_{\text{nc}}, \text{nc-function, locally bounded on slices}$$
$$\sup_n \sup_{r<1} \int_{(\text{bd}\Omega)_n} \text{tr}(f(rX)^* f(rX)) d\omega_n < \infty\}.$$

for $\Omega \in \{(\mathbb{D}^m)_{\text{nc}}, (\mathbb{B}^m)_{\text{nc}}\}$.

The Taylor-Taylor expansion around 0 for functions as above gives

$$f(X) = \sum_{l=0}^{\infty} \Big(\sum_{\substack{w \in \mathcal{F}_m \\ |w|=l}} X^w \cdot f_w \Big).$$

for $\mathcal{F}_m$ the free semigroup in m generators and
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- B.Collins 2003, B.Collins, P. Sniady 2006:
  Integration theory on $\mathcal{U}(n)$ for functions generated by

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  dependent on the difficult to handle " Weingarten function"

- Free Probabilities results:

  Haar unitaries with independent entries and constant matrices with limit distribution form an asympotically free family wrt

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- $H^2(\Omega)$ are inner-product spaces with the inner product

$$\langle f, g\rangle = \lim_{N\longrightarrow\infty} \lim_{r\longrightarrow 1^-} \int_{(\mathrm{bd}\Omega)_n} \mathrm{tr}\,(g(rX)^* f(rX))\, d\omega_N$$

  N.B.: the limit is not the supremum. For $m = 2$ and $f(X) = X_1X_2 + X_2X_1$,

$$\int_{(\mathrm{bd}\mathbb{D}^m_{\mathrm{nc}})_n} (f(rX)^* f(rX))\, d\omega_n = 2r^2(1 + \frac{1}{n^2})$$

- For each n, the boundary values $f(X) = \lim_{r\longrightarrow 1^-}$ exist a. e.
  The limit over r in the formula above con be replaced either by the $\sup$ or the integral of the boundary value.

- $\{X^w\}_{w\in\mathbb{F}_m}$ is a complete orthonormal system in $H^2((\mathbb{D}^m)_{\text{nc}})$; moreover $f_w = \langle f, X^w\rangle$ and $f = \sum_{w\in\mathcal{F}_m} f_w X^w$ in $H^2((\mathbb{D}^m)_{\text{nc}})$.

- $\{m^{\frac{|w|}{2}} X^w\}_{w\in\mathbb{F}_m}$ is a complete orthonormal system in $H^2((\mathbb{B}^m)_{\text{nc}})$; moreover, $f_w = \langle f, m^{|w|} X^w\rangle$ and $f = \sum_{w\in\mathcal{F}_m} f_w X^w$ in $H^2((\mathbb{B}^m)_{\text{nc}})$.

The spaces $H^2((\mathbb{D}^m)_{\text{nc}})$ and $H^2((\mathbb{B}^m)_{\text{nc}})$ are *not* isomorphic to some weighted l^2 spaces on $\mathcal{F}_m$. In fact, they are not complete.

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Consider the weighted l^2 spaces:

$$l^2_{(\mathbb{D}^m)_{\mathrm{nc}}} = \{(\alpha_w)_{w\in\mathcal{F}_m} : \sum_{w\in\mathcal{F}_m} |\alpha|^2 < \infty\}$$

$$l^2_{(\mathbb{B}^m)_{\mathrm{nc}}} = \{(\alpha_w)_{w\in\mathcal{F}_m} : \sum_{w\in\mathcal{F}_m} \frac{1}{m^{|w|}}|\alpha|^2 < \infty\}$$

For $\Omega \in \{(\mathbb{D}^m)_{\mathrm{nc}}, (\mathbb{B}^m)_{\mathrm{nc}}\}$, define

$$\Omega_{\mathrm{bd}} = \coprod_{n=1}^{\infty} \{X \in (\mathbb{C}^m)^{n\times n} : \{X^w\}_{w\in\mathcal{F}_w} \in l^2(\Omega) \otimes \mathbb{C}^{n\times n}\}$$

then

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- The completions $\overline{H^2}(\Omega)$ of $H^2(\Omega)$ can be identified with spaces of nc functions on Ω_{bd}:

$$\overline{H^2}(\Omega) = \{ f : \Omega_{\text{bd}} \longrightarrow \mathbb{C}_{\text{nc}} : f = \text{nc function with T-T expansion at 0}$$

$$f(X) = \sum_{w \in \mathcal{F}_m} f_w X^w$$

$$\text{for some } \{f_w\}_{w \in \mathcal{F}_m} \in l^2_\Omega \}$$

- $\Omega_n \cap (\Omega_{bd})_n$ consists of all $X \in \Omega_n$ such that the evaluation mapping

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- $\overline{H^2}(\Omega)$ are reproducing kernel Hilbert spaces, with the completely positive non-commutative kernels

$$K_{(\mathbb{D}^m)_{\rm nc}}(X,Y) = \sum_{l=0}^{\infty}[\sum_{|w|=l} X^w \otimes (Y^w)^*]$$

$$K_{(\mathbb{B}^m)_{\rm nc}}(X,Y) = \sum_{l=0}^{\infty}\frac{1}{m^l}[\sum_{|w|=l} X^w \otimes (Y^w)^*]$$

$$H^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}_{\text{nc}} : f \text{ nc function, } \sup_{X \in \Omega} \|f(X)\| \leq \infty\}$$

$$H^\infty(\widetilde{\Omega}) = \{f : \Omega \longrightarrow \mathbb{C}_{\text{nc}} : f \text{ nc function, } \sup_{X \in \Omega \cap \Omega_{\text{bd}}} \|f(X)\| \leq \infty\}$$

$$\mathcal{M}(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}_{\text{nc}} : f \text{ nc function and a bdd multiplier for } H^2(\Omega)\}$$

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$$\mathrm{Wg} : \mathbb{Z}_+ \times \bigcup_{n=1}^{\infty} S_n \longrightarrow \mathbb{C}$$

$$\mathrm{Wg}(N, \pi) = \int_{\mathcal{U}(N)} u_{1,1} \cdots u_{n,n} \overline{u_{1,\pi(1)}} \cdots \overline{u_{n,\pi(n)}} d\mathcal{U}_N(U)$$

analytic function in $\frac{1}{N}$, depending on the cycle decomposition of π:

$$\lim_{N \longrightarrow \infty} \frac{\mathrm{Wg}(N, \sigma)}{N^{2n - \#(\sigma)}} < \infty$$