Strong Asymptotic Freeness for Free Orthogonal Qauntum Groups

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Motivation: Polynomial Integrals Over O_N

Consider the $N \times N$ orthogonal group

$$O_N = \{g = [g_{ij}] \in M_N(\mathbb{C}) \,\big|\, g \text{ unitary } \& \; g^T = g^*\}$$

and the basic coordinate functions

$$v_{ij} \in C(O_N); \quad v_{ij}(g) = g_{ij} \qquad (g \in O_N, \; 1 \le i,j \le N).$$

Basic Problem

Given any polynomial function

$$f \in \mathrm{Pol}(O_N) := * - \mathrm{Alg}\big(v_{ij} : 1 \le i,j \le N\big) \subset C(O_N),$$

compute the Haar integral

$$h_{O_N}(f) = \int_{O_N} f(g)dg.$$

Equivalently, compute all joint moments of random variables

$$\{v_{ij}\}_{1 \le i,j \le N} \subset L^\infty(O_N, dg).$$

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- The need to compute polynomial integrals arises in many contexts: random matrices, (free) probability, QIT, physics...
- **Bad News:** Polynomial integrals over O_N are generally hard to explicitly compute (or even estimate).
- **Good News:** Often, one is interested in the polynomial integrals in the large N limit. In this setting, some simplifications occur. More precisely:

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Theorem (Attributed to many)

Let $W = \{w_{ij}\}_{i,j\in\mathbb{N}}$ be an i.i.d. array of $N(0,1)$ real Gaussian RVs over a probability space (Ω, P). Then the normalized coordinates $\{\sqrt{N}v_{ij}\}_{1\le i,j\le N}$ converge in distribution to $\mathcal{G}$. I.e., for any fixed k-tuples $i, j : [k] \to \mathbb{N}$,

$$\lim_{N\to\infty} h_{O_N}\left(\sqrt{N}v_{i(1)j(1)}\dots\sqrt{N}v_{i(k)j(k)}\right) = \int_\Omega w_{i(1)j(1)}\dots w_{i(k)j(k)}dP.$$

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We want to consider a "free version" of O_N, denoted by $\mathbb{F}O_N$. Define a universal unital $*$-algebra

$$\mathrm{Pol}(\mathbb{F}O_N) := *\!-\!\mathrm{Alg}\Big(\{u_{ij}\}_{1\leq i,j\leq N}\Big| U=[u_{ij}] \text{ is unitary } \& \ U=\overline{U}:=[u_{ij}^*]\Big)$$

$\mathrm{Pol}(\mathbb{F}O_N)$ is actually Hopf $*$-algebra with coproduct, counit and antipode determined by:

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik}\otimes u_{kj}$$
$$\epsilon(u_{ij}) = \delta_{ij}$$
$$S(u_{ij}) = u_{ji} \qquad (1\leq i,j\leq N).$$

$\mathrm{Pol}(\mathbb{F}O_N)$ together with Δ,ϵ,S yields a compact quantum group - the free orthogonal quantum group $\mathbb{F}O_N$.

Polynomial Integrals Over $\mathbb{F}O_N$

Since $\mathbb{F}O_N$ is a compact quantum group, there is a (faithful) Δ-invariant Haar state[1]:

$$h = h_{\mathbb{F}O_N} : \mathrm{Pol}(\mathbb{F}O_N) \to \mathbb{C}; \quad (h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = h(\cdot)1_{\mathrm{Pol}(\mathbb{F}O_N)}.$$

Thus we can talk about polynomial integrals over $\mathbb{F}O_N$:

For each $P \in \mathbb{C}\langle X_{ij} : 1 \leq i, j \leq N\rangle$ evaluate $h_{\mathbb{F}O_N}\big(P(\{u_{ij}\}_{1 \leq i,j \leq N})\big)$

GNS Construction: Put $L^2(\mathbb{F}O_N) := L^2(\mathrm{Pol}(\mathbb{F}O_N), h)$ and

$$L^\infty(\mathbb{F}O_N) = \mathrm{Pol}(\mathbb{F}O_N)'' \subseteq \mathcal{B}(L^2(\mathbb{F}O_N)).$$

The von Neumann algebra $L^\infty(\mathbb{F}O_N)$ is completely determined by the above polynomial integrals.

[1] h is tracial.

Integrals via Weingarten Calculus

- Fix $k \in \mathbb{N}$ and let $NC_2(k) =$ set of non-crossing pairings of $[k] = \{1, 2, \ldots, k\}$.
- For each $i : [k] \to \mathbb{N}$, $\pi \in NC_2(k)$, put

$$\delta_\pi(i) = \begin{cases} 1 & \text{if } i \text{ is constant on each block of } \pi \\ 0 & \text{otherwise} \end{cases}$$

  Example: $\delta_{\sqcup\,\sqcup}(i) = 1$ iff $i(1) = i(2) \& i(3) = i(4)$.
- Finally define a matrix $W_{k,N} \in M_{NC_2(k) \times NC_2(k)}(\mathbb{C})$ by

$$W_{k,N}^{-1} = [N^{\#(\pi \vee \sigma)}]_{\pi, \sigma \in NC_2(k)}.$$

Theorem (Banica-Collins '07)

$$h_{\mathbb{F}O_N}(u_{i(1)j(1)} \ldots u_{i(k)j(k)}) = \sum_{\pi, \sigma \in N_2(k)} \delta_\pi(i)\delta_\sigma(j) W_{k,N}(\pi, \sigma).$$

Observation: $W_{k,N}(\pi, \sigma) = N^{-k/2}(\delta_{\pi,\sigma} + O(N^{-1}))$.

Asymptotic Freeness

Corollary (Banica-Collins '07)

Let $S = \{s_{ij}\}_{i,j \in \mathbb{N}}$ *be a free semicircular system in a finite von Neumann algebra* (M, τ) *Then the the normalized coordinates* $S_N = \{\sqrt{N} u_{ij}\}_{1 \leq i,j \leq N}$ *converge in distribution to* S*.*

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Proof.

Use the asymptotics $W_{k,N}(\pi,\sigma) = N^{-k/2}(\delta_{\pi,\sigma} + O(N^{-1}))$:

$$
\begin{aligned}
&h_{\mathbb{F}O_N}(\sqrt{N}u_{i(1)j(1)}\dots\sqrt{N}u_{i(k)j(k)}) \\
&= N^{k/2}\sum_{\pi,\sigma\in N_2(k)}\delta_\pi(i)\delta_\sigma(j)W_{k,N}(\pi,\sigma) = \sum_{\pi\in N_2(k)}\delta_\pi(i)\delta_\pi(j) + O(N^{-1}) \\
&= \#\{\pi \mid i,j \text{ constant on blocks of } \pi\} + O(N^{-1}) \\
&= \tau(s_{i(1)j(1)}\dots s_{i(k)j(k)}) + O(N^{-1}).
\end{aligned}
$$

□

Thus the generators $\{\sqrt{N}u_{ij}\}_{1\leq i,j\leq N}$ of the von Neumann algebra $L^\infty(\mathbb{F}O_N)$ are asymptotically free and semicircular.

Strong Asymptotic Freeness

Question

Can we say anything more about the mode of convergence of

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Theorem (B. '13)

S_N is strongly asymptotically free and semicircular. I.e., for any non-commutative polynomial $P \in \mathbb{C}\langle X_{ij} : i,j \in \mathbb{N}\rangle$,

$$h_{\mathbb{F}O_N}(P(S_N)) \longrightarrow \tau(P(S)) \quad \& \quad \|P(S_N)\|_{L^\infty(\mathbb{F}O_N)} \longrightarrow \|P(S)\|_{L^\infty(M,\tau)}.$$

Remark: Using standard C*-algebraic techniques, a similar result also holds for matrix-valued polynomials $P \in M_k(\mathbb{C}) \otimes \mathbb{C}\langle X_{ij} : i,j \in \mathbb{N}\rangle$.

The One Variable Case: Superconvergence of $\sqrt{N}u_{11}$

The one-variable version of the above theorem was already known.

Theorem (Banica-Collins-Zinn Justin '09)

Let μ_N be the spectral measure of $\sqrt{N}u_{11} \in L^\infty(\mathbb{F}O_N)$. Then

- *μ_N is atomless and $\mathrm{supp}\mu_N = \Big[-2\sqrt{\frac{N}{N+2}}, 2\sqrt{\frac{N}{N+2}}\Big]$.*
- *$\frac{d\mu_N}{dt}$ is analytic on $\Big(-2\sqrt{\frac{N}{N+2}}, 2\sqrt{\frac{N}{N+2}}\Big)$ and converges uniformly to $\frac{\sqrt{4-t^2}}{2\pi} = \frac{d(\text{semicircle law})}{dt}$.*

The proof involves modeling u_{11} as a certain variable over Woronowicz' $SU_q(2)$ quantum group ($N = -q - q^{-1}$), and exploiting the structure there.

For the Multivariate Setting...

We work with moments. Fix $P \in \mathbb{C}\langle X_{ij} : i, j \in \mathbb{N}\rangle$. Recall that

$$\|P(S_N)\|_{L^\infty(\mathbb{F}O_N)} = \lim_{q\to\infty} \|P(S_N)\|_{L^q(\mathbb{F}O_N)},$$

$$\lim_{N\to\infty} \|P(S_N)\|_{L^q(\mathbb{F}O_N)} = \|P(S)\|_{L^q(M,\tau)} \qquad (q \in 2\mathbb{N}).$$

Consequently, $\liminf_{N\to\infty} \|P(S_N)\|_{L^\infty(\mathbb{F}O_N)} \geq \|P(S)\|_{L^\infty(M,\tau)}$.

We want: $$\limsup_{N\to\infty} \|P(S_N)\|_{L^\infty(\mathbb{F}O_N)} \leq \|P(S)\|_{L^\infty(M,\tau)}.$$

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Proposition (Uniform L^q-L^∞ estimates for $\mathbb{F}O_N$)

For any $\epsilon > 0$, there is a $q = q(P, \epsilon) > 0$ such that

$$\|P(S_N)\|_{L^\infty(\mathbb{F}O_N)} \leq (1+\epsilon)\|P(S_N)\|_{L^q(\mathbb{F}O_N)} \qquad \textit{UNIFORMLY in } N.$$

Letting $N \to \infty, q \to \infty, \epsilon \to 0$, we get

$$\limsup_{N\to\infty} \|P(S_N)\|_{L^\infty(\mathbb{F}O_N)} \leq \|P(S)\|_{L^\infty(M,\tau)}.$$

Uniform L^q-L^∞ estimate

We use Vergnioux's property of rapid decay for the dual quantum groups $\widehat{\mathbb{F}O_N}$.

- Inductively define subspaces $\{H_k\}_{k\geq 0}$ of $\mathrm{Pol}(\mathbb{F}O_N)$ where

$$H_0(N) = \mathbb{C}1, \quad H_1(N) = \mathrm{span}\{u_{ij}\}_{1\leq i,j\leq N}$$

and $$H_k(N) = \mathrm{span}\{H_1(N)H_{k-1}(N)\} \ominus H_{k-2}(N) \qquad (k \geq 2).$$

- Peter-Weyl/"Fock" decomposition (Banica '95):

$$\mathrm{Pol}(\mathbb{F}O_N) = L^2 - \bigoplus_{k\geq 0} H_k(N).$$

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Theorem ("Property RD" Vergnioux '07)

Let $P_k : Pol(\mathbb{F}O_N) \to H_k(N)$ be the $\perp$-projection. Then there is a constant $D_N > 1$ (only depending on N) such that for any $k, l, n \in \mathbb{N}$

$$\|P_k(xy)\|_{L^2} \leq D_N \|x\|_{L^2} \|y\|_{L^2} \qquad (x \in H_l(N),\ y \in H_n(N)).$$

For our polynomial $P(S_N)$, we have

$$P(S_N) \in \bigoplus_{k=0}^{\deg P} H_k(N)$$

and property RD + some book-keeping implies

$$\|P(S_N)\|_{L^\infty} \leq D_N(\deg P + 1)^{3/2} \|P(S_N)\|_{L^2}.$$

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Replacing P by $(P^*P)^m$ ($m \in \mathbb{N}$), we get an L^{4m}-L^∞ estimate:

$$\|(P^*P)^m(S_N)\|_{L^\infty} \leq D_N(\underbrace{\deg(P^*P)^m}_{\leq 2m\deg P}+1)^{3/2}\|(P^*P)^m(S_N)\|_{L^2}$$

$$\Longrightarrow \|P(S_N)\|_{L^\infty} \leq D_N^{1/2m}(2m\deg P + 1)^{3/4m}\|P(S_N)\|_{L^{4m}} \qquad (m \in \mathbb{N}).$$

To conclude, one just needs to show that

$$\lim_{m\to\infty} D_N^{1/2m}(2m\deg P + 1)^{3/4m} = 1 \qquad \text{uniformly in } N.$$

This is established by proving that $\{D_N\}_{N\geq 3}$ is bounded.

Some Applications

- Since strong convergence is stable with respect to taking reduced free products (Skoufranis '12), get strong asymptotic freeness for the free unitary quantum groups $\mathbb{F}U_N$ from $\mathbb{F}O_N$.
- Can deduce well known L^2-L^∞ inequalities for polynomials in semicircular systems from the corresponding ones for $L^\infty(\mathbb{F}O_N)$ given by property RD.