Orbital Free Entropy and its dimension counterpart

Yoshimichi UEDA

Kyushu University

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Voiculescu proposed it at the end of the part 6 of his series of papers on free entropy. Then Hiai and I studied its details.

- Voiculescu, The analogue of entropy and of Fisher's information measure in free probability theory, VI: Liberation and mutual free information, Adv. Math., 149 (1999), 101–166.
- Hiai and Petz, Large deviation for functions of two random projection matrices, Acta Sci. Math. (Szeged), 72(2006), 581–609.
- Hiai and U., Notes on microstate free entropy of projections, Publ. RIMS, 44 (2008), 49–89.
- Hiai and U., A log-Sobolev type inequality for free entropy of two projections, AIHP, 45 (2009), 239–249.

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- $\mathbb{R}^+_{\geq} = \{(\lambda_1, \dots, \lambda_N) \mid \lambda_i \geq \lambda_{i+1}\}$, $\mathbb{R}^+_{>} = \{(\lambda_1, \dots, \lambda_N) \mid \lambda_i \gneqq \lambda_{i+1}\}$
- $\mathrm{U}(N)/\mathbb{T}^N$ with the diagonals $\mathbb{T}^N$ in $\mathrm{U}(N)$.

The surjection (the (inverse) diagonalization map)

$$\Phi_N : ([U], \lambda) \in (\mathrm{U}(N)/\mathbb{T}^N) \times \mathbb{R}^N_{\geq} \mapsto U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^* \in M_N(\mathbb{C})^{sa}$$

becomes a measure space isomorphism with respect to

$$C_N \,\mathbf{Haar} \otimes \Big(\prod_{i<j} (\lambda_i - \lambda_j)^2 \, d\lambda\Big), \quad \mathbf{Leb.}$$

The surjection Φ_N is injective on $(\mathrm{U}(N)/\mathbb{T}^N) \times \mathbb{R}^N_{>}$.

Matricial microstates (Voiculescu)

Let $X_1, \dots, X_n$ be non-commutative self-adjoint random variables in a tracial W^*-prob. space $(\mathcal{M}, \tau)$. The space of matricial microstates

$$\Gamma_R(X_1, \dots, X_n \,; N, m, \varepsilon) \quad \text{possibly } R = \infty$$

is the subset of $(M_N(\mathbb{C})^{sa})^n$ consisting of all (A_i) such that

- the operator norm $\|A_i\|_\infty \leq R$ (nothing when $R = \infty$),
- for every $1 \leq i_1, \dots, i_k \leq n$ and $1 \leq k \leq m$

$$\left| \frac{1}{N} \mathrm{Tr}(A_{i_1} \cdots A_{i_k}) - \tau(X_{i_1} \cdots X_{i_k}) \right| < \varepsilon.$$

Write

$$\chi_R(X_1, \dots, X_n \,; N, m, \varepsilon) := \log \left(\mathrm{Leb}^{\otimes n} \left(\Gamma_R(X_1, \dots, X_n \,; N, m, \varepsilon) \right) \right).$$

Then

$$\chi_R(X_1,\dots,X_n) := \lim_{\substack{m\to\infty \\ \varepsilon\searrow 0}} \limsup_{N\to\infty} \left\{\frac{1}{N^2}\chi_R(X_1,\dots,X_n\,;N,m,\varepsilon) + \frac{n}{2}\log N\right\}$$

is a constant function in $R \in [\max \|X_i\|_\infty, +\infty]$ and called the free entropy $\chi(X_1,\dots,X_n)$ of $X_1,\dots,X_n$.

Then

$$\chi_R(X_1,\dots,X_n)$$
$$:= \lim_{\substack{m\to\infty \\ \varepsilon\searrow 0}} \limsup_{N\to\infty} \left\{\frac{1}{N^2}\chi_R(X_1,\dots,X_n\,;N,m,\varepsilon) + \frac{n}{2}\log N\right\}$$

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The free entropy is a right quantity playing a role of “entropy” in free probability; for example,

- $X_1,\dots,X_n$ freely independent $\Longrightarrow$ $\chi(X_1,\dots,X_n) = \chi(X_1) + \cdots + \chi(X_n)$;
- ($\Leftarrow$) also holds under $\chi(X_1,\dots,X_n) > -\infty$.

Orbital microstates – 1st approach (Hiai, Miyamoto and U.)

The diagonalization map Φ_N induces a continuous surjection:

$$\Phi_{n,N} : (\mathrm{U}(N)/\mathbb{T}^N)^n \times (\mathbb{R}^N_{\geq})^n \to (M_N(\mathbb{C})^{sa})^n.$$

Then the set of “orbital microstates”

$$\tilde{\Gamma}_{\mathrm{orb},R}(X_1, \ldots, X_n ; N, m, \varepsilon)$$

is defined to be the projection of

$$\Phi^{-1}_{n,N}(\Gamma_R(X_1, \ldots, X_n ; N, m, \varepsilon))$$

onto $(\mathrm{U}(N)/\mathbb{T}^N)^n$.

The volume

$$(\mathrm{Haar}_{\mathrm{U}(N)/\mathbb{T}^N})^{\otimes n}\Big(\tilde{\Gamma}_{\mathrm{orb},R}(X_1,\dots,X_n\,;N,m,\varepsilon)\Big)$$

is nothing but the usual Haar probablity measure of the space of all the n-tuples $(U_i) \in \mathrm{U}(N)^n$ such that $\exists D_i$, $N \times N$ diagonal real matrices whose entries are not increasing and

$$(U_i D_i U_i^*) \in \Gamma_R(X_1,\dots,X_n\,;N,m,\varepsilon).$$

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We call this latter subset of $\mathrm{U}(N)^n$ the space of orbital microstates and denote it by $\Gamma_{\mathrm{orb},R}(X_1,\dots,X_n\,;N,m,\varepsilon)$. Write

$$\begin{aligned}&\chi_{\mathrm{orb},R}(X_1,\dots,X_n\,;N,m,\varepsilon)\\ &:= \log\Big((\mathrm{Haar}_{\mathrm{U}(N)})^{\otimes n}\Big(\Gamma_{\mathrm{orb},R}(X_1,\dots,X_n\,;N,m,\varepsilon)\Big)\Big).\end{aligned}$$

Orbital free entropy – 1st approach (Hiai, Miyamoto and U.)

Then

$$\chi_{\mathrm{orb},R}(X_1,\dots,X_n) := \lim_{\substack{m\to\infty \\ \varepsilon\nearrow 0}} \limsup_{N\to\infty} \frac{1}{N^2}\chi_{\mathrm{orb},R}(X_1,\dots,X_n\,;N,m,\varepsilon)$$

is independent of the choice of $R \in [\max \|X_i\|_\infty, +\infty]$, and we call this the orbital free entropy $\chi_{\mathrm{orb}}(X_1,\dots,X_n)$.

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Proposition ([Hiai, Miyamoto and U., IJM, 2009])

$$\chi(X_1,\dots,X_n) = \chi_{\mathrm{orb}}(X_1,\dots,X_n) + \sum_{i=1}^{n} \chi(X_i)$$

Remember: $H(X,Y) = -I(X,Y) + H(X) + H(Y)$.

Orbital free entropy – 2nd approach (Hiai, Miyamoto and U.)

Free entropy adapted to projections uses:

- deteministic $P_i(N) = P_i(N)^* = P_i(N)^2$ with $\mathbf{rk}(P_i(N))/N \to \tau(P_i)$;
- $\Gamma_{\mathrm{proj}}(P_1, \ldots, P_n\,; N, m, \delta)$
  $= \{(U_i) \in \mathrm{U}(N)^n \,|\, (U_i P_i(N) U_i^*) \in \Gamma_1(P_1, \ldots, P_n\,; N, m, \varepsilon)\}$.

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Another formulation: the space of orbital microstates

$$\Gamma_{\mathrm{orb}}(X_1, \dots, X_n : (\xi_i(N))\,; N, m, \varepsilon)$$

are defined to be all the n-tuples of unitaries $(U_i) \in \mathrm{U}(N)^n$ such that $(U_i \xi_i(N) U_i^*) \in \Gamma_\infty(X_1, \dots, X_n; N, m, \varepsilon)$, where the $\xi_i(N)$ are the deterministic self-adjoint matrices such that each $\xi_i(N)$ converges to X_i in moments.

Define

$$\chi_{\mathrm{orb}}(X_1,\dots,X_n : (\xi_i(N))\,;N,m,\varepsilon)$$
$$:= \log\Big((\mathrm{Haar}_{\mathrm{U}(N)})^{\otimes n}\big(\Gamma_{\mathrm{orb}}(X_1,\dots,X_n : (\xi_i(N))\,;N,m,\varepsilon)\big)\Big),$$

and we can prove that

$$\chi_{\mathrm{orb}}(X_1,\dots,X_n : (\xi_i(N)))$$
$$:= \lim_{\substack{m\to\infty \\ \varepsilon\nearrow 0}} \limsup_{N\to\infty} \frac{1}{N^2}\chi_{\mathrm{orb}}(X_1,\dots,X_n : (\xi_i(N))\,;N,m,\varepsilon)$$

coincides with the previous orbital free entropy $\chi_{\mathrm{orb}}(X_1,\dots,X_n)$, or other words, it does never depend on the choice of approximation $\xi_i(N)$.

Orbital free entropy for HF multivariables (Hiai, Miyamoto and U.)

Replace:

$$X_i \rightsquigarrow \mathbb{X}_i = \{X_{i1}, \ldots, X_{il_i}\} \text{ with HF (=hyperfinite) } W^*(\mathbb{X}_i),$$
$$\xi_i(N) \rightsquigarrow \Xi_i(N) = \{\xi_{i1}(N), \ldots, \xi_{il_i}(N)\}.$$

We define

$$\begin{aligned}
&\Gamma_{\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n : (\Xi_i(N)) ; N, m, \varepsilon) \\
&\quad = \{(U_i) \in \mathrm{U}(N)^n \mid (U_i \Xi_i(N) U_i^*) \in \Gamma_\infty(\mathbb{X}_1, \ldots, \mathbb{X}_n ; N, m, \varepsilon)\}, \\
&\chi_{\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n : (\Xi_i(N)) ; N, m, \varepsilon) \\
&\quad := \log\Big((\mathrm{Haar}_{\mathrm{U}(N)})^{\otimes n}\Big(\Gamma_{\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n : (\Xi_i(N)) ; N, m, \varepsilon)\Big)\Big).
\end{aligned}$$

We can prove that

$$\chi_{\mathrm{orb}}(\mathbb{X}_1,\dots,\mathbb{X}_n) := \lim_{\substack{m\to\infty \\ \varepsilon\nearrow 0}} \limsup_{N\to\infty} \frac{1}{N^2}\chi_{\mathrm{orb}}(\mathbb{X}_1,\dots,\mathbb{X}_n : (\Xi_i(N))\,;N,m,\varepsilon)$$

does never depend on the choice of approximation $\Xi_i(N)$. We call this the orbital free entropy of HF random multivariables $\mathbb{X}_1,\dots,\mathbb{X}_n$.

The key: Jung's characterization of hyperfiniteness in terms of matricial microstates.

Properties of Orbital Free Entropy

Let X_i be (non-commutative) random variables, and $\mathbb{X}_i, \mathbb{Y}_i$ be HF random multivariables. Let U_i be unitary random variables.

Proposition ([Hiai, Miyamoto and U., IJM, 2009])

- $\chi(X_1, \dots, X_n) = \chi_{\mathrm{orb}}(X_1, \dots, X_n) + \sum_{i=1}^n \chi(X_i)$;
- $\chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) = 0 \Longleftrightarrow \mathbb{X}_1, \dots, \mathbb{X}_n$ freely independent;
- $W^*(\mathbb{X}_i) = W^*(\mathbb{Y}_i) \Longrightarrow \chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) = \chi_{\mathrm{orb}}(\mathbb{Y}_1, \dots, \mathbb{Y}_n)$;
- $\mathbb{X}_i \subseteq W^*(\mathbb{Y}_i) \Longrightarrow \chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) \leq \chi_{\mathrm{orb}}(\mathbb{Y}_1, \dots, \mathbb{Y}_n)$;
- all U_i are freely independent of the $\mathbb{X}_i$ and $\chi_u(U_i) > -\infty \Longrightarrow$ $\chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) \leq \chi_{\mathrm{orb}}(U_1\mathbb{X}_1U_1^*, \dots, U_n\mathbb{X}_nU_n^*)$.

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The proof of the 5th property uses the conditional variant of χ_{orb};

$$\chi_{\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n : Z_1, \ldots, Z_{n'})$$

defined as same as the conditional variant of free entropy.

Let P, Q be two projections in a W^*-prob. space $(\mathcal{M}, \tau)$. Then PQP or equivalently QPQ gives a prob. meas. ν on $[0,1]$ via spectral decomp. with τ. Then

$$\begin{aligned}\chi_{\mathrm{orb}}(P,Q) &= \chi_{\mathrm{proj}}(P,Q) \\ &= \int\int_{(0,1)^2} \log|x-y|\,\nu\otimes\nu(dx,dy) \\ &+ |\tau(P)-\tau(Q)| \int_{(0,1)} \log x\,\nu(dx) \\ &+ |\tau(P)+\tau(Q)-1| \int_{(0,1)} \log(1-x)\,\nu(dx) + C\end{aligned}$$

when P, Q are in generic position; otherwise $= -\infty$.

Classical mutual information

$$I(X,Y) = \int\int \log \frac{d\mu_{(X,Y)}}{d(\mu_X \otimes \mu_Y)}\, d\mu_{(X,Y)} \quad \text{for r.v.'s } X, Y$$

$$= \int\int \log \frac{d\mu_{(X,Y)}}{dx\,dy}\, d\mu_{(X,Y)}$$

$$- \int\int \log \frac{d\mu_X}{dx} d\mu_{(X,Y)} - \int\int \log \frac{d\mu_Y}{dx} d\mu_{(X,Y)}$$

$$= -H(X,Y) + H(X) + H(Y).$$

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Voiculescu defined the free mutual Fisher information

$$\varphi^*(A_1\,;\ldots;A_n)$$

for (unital) $*$-subalgebras $A_1,\ldots,A_n$ of $\mathcal{M}$ by developping appropriate derivation theory like the approach χ^* of free entropy, and then the free mutual information

$$i^*(A_1\,;\ldots;A_n) = \frac{1}{2}\int_0^\infty \varphi^*(U_1(1)A_1U_1(t)^*\,;\ldots;U_n(t)A_nU_n(t)^*)\,dt$$

with freely independent family $U_i(t)$ of unitary BMs.

Free mutual information (Voiculescu)

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with freely independent family $U_i(t)$ of unitary BMs. This enjoys almost all the properties that $-\chi_{\mathrm{orb}}$ has except the exact relation to χ itself.

A test to $i^* = -\chi_{\mathrm{orb}}$

Conjecture (or hope ?)

$$i^*(W^*(\mathbb{X}_1)\,;\ldots;W^*(\mathbb{X}_n)) = -\chi_{\mathrm{orb}}(\mathbb{X}_1,\ldots,\mathbb{X}_n).$$

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Theorem ([Izumi and U., arXiv:1306.5372])

If P, Q are projection in $(\mathcal{M},\tau)$ with $\tau(P) = \tau(Q) = 1/2$, then $i^*(W^*(P)\,;W^*(Q)) = -\chi_{\rm orb}(P,Q)$ holds possibly with $+\infty = +\infty$.

Comments:

- Collins and Kemp [arXiv:1211.6037] gave the same result under a restricted assumption.
- The result should be regarded as an orbital-counterpart of $\chi^*(X) = \chi(X)$ for single X due to Voiculescu.
- We have a partial result without assuming $\tau(P) = \tau(Q) = 1/2$. The 'perfect' subordination result was obtained based on Loewner (–Kufarev) equations. The remaining issue is some analysis of Loewner equations.

Beyond HF – 3rd approach ([Biane and Dabrowski, Adv. Math., 2013])

Replacing

- matricial microstates $\rightsquigarrow$ "random microstates";
- Leb. meas. $\rightsquigarrow$ classical entropy H,

Biane and Dabrowski introduced a concavification $\tilde{\chi}$ of free entropy χ. In the same sprit, they introduced new approaches to the orbital free entropy $\chi_{\mathbf{orb}}$, whose definitions involve directly classical mutual information.

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Biane and Dabrowski introduced a concavification $\tilde{\chi}$ of free entropy χ. In the same sprit, they introduced new approaches to the orbital free entropy $\chi_{\mathbf{orb}}$, whose definitions involve directly classical mutual information. Their variants of $\chi_{\mathbf{orb}}$ don't require the assumption that $W^*(\mathbb{X}_i)$ are HF (positive side), but can be proved to coincide with the original $\chi_{\mathbf{orb}}$ only when $W^*(\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n)$ is a factor (negative side for me).

$$\Psi_N : (U_i, (\mathbb{A}_i)) \in \mathrm{U}(N)^n \times \Big(\prod_{i=1}^{n} (M_N(\mathbb{C})^{sa})^{l_i} \Big)$$
$$\longrightarrow (U_i \mathbb{A}_i U_i^*) \in \prod_{i=1}^{n} (M_N(\mathbb{C})^{sa})^{l_i}.$$

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Key lemma

For any prob. meas. μ on $\prod_{i=1}^n (M_N(\mathbb{C})^{sa})^{l_i}$

$$(\mathrm{Haar} \otimes \mu)(\Psi_N^{-1}(\Gamma_R(\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n\,; N, m, \varepsilon)))$$
$$= \int_{\prod_{i=1}^n \Gamma_R(\mathbb{X}_i\,; N,m,\varepsilon)} \mathrm{Haar}(\Gamma_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n : (\mathbb{A}_i)\,; N, m, \varepsilon))\, \mu(d(\mathbb{A}_i)).$$

$$\chi_{\mathrm{orb},R}(\mathbb{X}_1,\dots,\mathbb{X}_n : \mu\,; N,m,\varepsilon)$$
$$:= \log\Big((\mathrm{Haar}\otimes\mu)(\Psi_N^{-1}(\Gamma_R(\mathbb{X}_1\sqcup\cdots\sqcup\mathbb{X}_n\,; N,m,\varepsilon)))\Big),$$
$$\chi_{\mathrm{orb}}(\mathbb{X}_1,\dots,\mathbb{X}_n : (\mathbb{A}_i)\,; N,m,\varepsilon)$$
$$= \log\Big(\mathrm{Haar}(\Gamma_{\mathrm{orb}}(\mathbb{X}_1,\dots,\mathbb{X}_n : (\mathbb{A}_i)\,; N,m,\varepsilon))\Big).$$

Then

$$\lim_{m,\varepsilon}\limsup_{N}\frac{1}{N^2}\sup_{\mu}\chi_{\mathrm{orb},R}(\mathbb{X}_1,\dots,\mathbb{X}_n : \mu\,; N,m,\varepsilon)$$
$$= \lim_{m,\varepsilon}\limsup_{N}\frac{1}{N^2}\sup_{(\mathbb{A}_i)}\chi_{\mathrm{orb}}(\mathbb{X}_1,\dots,\mathbb{X}_n : (\mathbb{A}_i)\,; N,m,\varepsilon)$$

is independent of the choice of $R \in [\max\|X_{ij}\|_\infty, +\infty]$ and agrees with the previous χ_{orb}.

Beyond HF – 4th approach: observation

Assume: all $\mathbb{X}_i$ have f.d.a., or other words, $W^*(\mathbb{X}_i) \subset R^\omega$.

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Let $\nu(N, m, \varepsilon)$ be the normalized Leb. meas. on $\prod_{i=1}^n \Gamma_R(\mathbb{X}_i; N, m, \varepsilon)$ with $R \geq \max \|X_{ij}\|_\infty$. Then

$$\begin{aligned}
&\sup_\mu \chi_{\mathrm{orb},R}(\mathbb{X}_1, \ldots, \mathbb{X}_n : \mu ; N, m, \varepsilon) \\
&\geq \chi_{\mathrm{orb},R}(\mathbb{X}_1, \ldots, \mathbb{X}_n : \nu(N, m, \varepsilon) ; N, m, \varepsilon) \\
&= \chi_R(\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n; N, m, \varepsilon) - \sum_{i=1}^n \chi_R(\mathbb{X}_i ; N, m, \varepsilon).
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$$\begin{aligned}
&\sup_\mu \chi_{\mathrm{orb},R}(\mathbb{X}_1,\dots,\mathbb{X}_n : \mu\,; N,m,\varepsilon) \\
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\end{aligned}$$

Proposition ([U., IUMJ, to appear])

$$\chi(\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n) \leq \chi_{\mathrm{orb}}(\mathbb{X}_1,\dots,\mathbb{X}_n) + \sum_{i=1}^n \chi(\mathbb{X}_i).$$

Proposition ([U., IUMJ, to appear])

- $\chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) = 0 \iff \mathbb{X}_1, \dots, \mathbb{X}_n$ freely independent.
- $W^*(\mathbb{X}_i) \subseteq W^*(\mathbb{Y}_i) \implies \chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) \leq \chi_{\mathrm{orb}}(\mathbb{Y}_1, \dots, \mathbb{Y}_n)$.
- $W^*(\mathbb{X}_i) = W^*(\mathbb{Y}_i) \implies \chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) = \chi_{\mathrm{orb}}(\mathbb{Y}_1, \dots, \mathbb{Y}_n)$.
- If all U_i are freely independent of the $\mathbb{X}_i$ and $\chi_u(U_i) > -\infty$, then

$$\begin{aligned}\chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) &\leq \chi_{\mathrm{orb}}(U_1\mathbb{X}_1U_1^*, \dots, U_n\mathbb{X}_nU_n^* : U_1, \dots, U_n) \\ &\leq \chi_{\mathrm{orb}}(U_1\mathbb{X}_1U_1^*, \dots, U_n\mathbb{X}_nU_n^*).\end{aligned}$$

Comment: A big issue is to prove the opposite inequality:

$$\chi(\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n) \geq \chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) + \sum_{i=1}^{n} \chi(\mathbb{X}_i).$$

Orbital Free Entropy Dimension (Hiai, Miyamoto and U.)

The orbital free entropy dimension $\delta_{0,\mathrm{orb}}(\mathbb{X}_1,\dots,\mathbb{X}_n)$ is defined to be

$$\limsup_{t\searrow 0}\frac{\chi_{\mathrm{orb}}(U_1(t)\mathbb{X}_1U_1(t)^*,\dots,U_n(t)\mathbb{X}_nU_n(t)^* : U_1(t),\dots,U_n(t))}{|\log\sqrt{t}|},$$

where the $U_i(t)$'s are freely independent, free unitary BMs, freely independent of the $\mathbb{X}_i$'s.

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where the $U_i(t)$'s are freely independent, free unitary BMs, freely independent of the $\mathbb{X}_i$'s.

Idea: Replace

$$\chi \text{ and } X+\sqrt{t}S \quad\rightsquigarrow\quad \chi_{\mathrm{orb}} \text{ and } X\mapsto U(t)XU(t)^*$$

in the definition of the original δ_0.

Properties of orbital free entropy dimension

Proposition ([HMU, IJM, 2009],[U, IUMJ, to appear])

- $\delta_{0,\mathrm{orb}} \leq 0$.
- $\delta_{0,\mathrm{orb}}(\mathbb{X}) = 0$ if $\mathbb{X}$ has f.d.a.
- Monotonicity, i.e., $\mathbb{Y}_i \subset W^*(\mathbb{X}_i)$ for all i implies

  $$\delta_{0,\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n) \leq \delta_{0,\mathrm{orb}}(\mathbb{Y}_1, \ldots, \mathbb{Y}_n).$$

  Hence $\delta_{0,\mathrm{orb}}$ does depend only on the $W^*(\mathbb{X}_i)$.
- $\chi_{\mathrm{orb}} > -\infty$ implies $\delta_{0,\mathrm{orb}} = 0$.
- If $\mathbb{X}_1$ and $\mathbb{X}_2 \sqcup \cdots \sqcup \mathbb{X}_n$ are freely independent and if $\mathbb{X}_1$ has f.d.a., then

  $$\delta_{0,\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n) = \delta_{0,\mathrm{orb}}(\mathbb{X}_2, \ldots, \mathbb{X}_n).$$

Proposition ([HMU, IJM 2009],[U.,IUMJ to appear])

Jung's packing formulation still works for $\delta_{0,\mathrm{orb}}$.

Important Formula

Theorem ([HMU, IJM '09])

We have

$$\delta_0(\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n) = \delta_{0,\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n) + \sum_{i=1}^{n} \delta_0(\mathbb{X}_i)$$

under the assumption that all $W^*(\mathbb{X}_i)$ are HF.

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Comments:

- We used Jung's precise result on δ_0 in the HF case.
- A big issue is again whether or not the formula still holds true without assuming that all $W^*(\mathbb{X}_i)$ are HF.

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- We used Jung's precise result on δ_0 in the HF case.
- A big issue is again whether or not the formula still holds true without assuming that all $W^*(\mathbb{X}_i)$ are HF. I don't know even whether or not ($\leq$) holds.

Theorem ([HMU, IJM, 2009])

Let $\mathbb{X}_i = \{X_{i1}, \dots, X_{il_i}\}$ and $\mathbb{X}_i^{(k)} = \{X_{i1}^{(k)}, \dots, X_{il_i}^{(k)}\}$ $(1 \le i \le n,\ k = 1, 2 \dots)$ be tuples of self-adjoint elements in a W^*-probability space such that

- every $W^*(\mathbb{X}_i)$ is HF $(1 \le i \le n)$,
- $\mathbb{X}_i^{(k)} \subseteq W^*(\mathbb{X}_i)$
- $X_{ij}^{(k)} \longrightarrow X_{ij}$ strongly.

Then:

$$\delta_0(\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_m) \le \liminf_{k\to\infty} \delta_0(\mathbb{X}_1^{(k)} \sqcup \cdots \sqcup \mathbb{X}_m^{(k)}).$$

Proof

Firstly prove: $\delta_0(\mathbb{X}_i) \leq \liminf_{k\to\infty} \delta_0(\mathbb{X}_i^{(k)})$ if $\mathbb{X}^{(k)} \to \mathbb{X}$ strongly.
(Jung's result was used at this point !)

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$$
\begin{aligned}
&\liminf_{k\to\infty} \delta_0(\mathbb{X}_1^{(k)} \sqcup \cdots \sqcup \mathbb{X}_m^{(k)}) \\
&\qquad = \liminf_{k\to\infty} \left\{\delta_{0,\mathrm{orb}}(\mathbb{X}_1^{(k)}, \ldots, \mathbb{X}_m^{(k)}) + \sum_{i=1}^m \delta_0(\mathbb{X}_i^{(k)})\right\} \\
&\qquad \ge \liminf_{k\to\infty} \left\{\delta_{0,\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_m) + \sum_{i=1}^m \delta_0(\mathbb{X}_i^{(k)})\right\} \\
&\qquad \ge \delta_{0,\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_m) + \sum_{i=1}^m \liminf_{k\to\infty} \delta_0(\mathbb{X}_i^{(k)}) \\
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Let $\mathbb{X}_i$ be HF and $U_i(t)$ be freely independent, free unitary BMs, freely independent of the $\mathbb{X}_i$'s. If $\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n$ has f.d.a., then

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as long as $t \gneq 0$.

Another application of the formula

Let $\mathbb{X}_i$ be HF and $U_i(t)$ be freely independent, free unitary BMs, freely independent of the $\mathbb{X}_i$'s. If $\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n$ has f.d.a., then

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Question:
What properties do $W^*(U_1(t)M_1U_1(t)^* \sqcup \cdots \sqcup U_n(t)M_nU_n(t)^*)$ and $M_1 \star \cdots \star M_n$ have in common ?

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Remark: Dabrowski and Ioana [arXiv:1212.6425] and also Dabrowski (in preparation) recently proved some results in the direction (as we saw in Dabrowski's talk).

Remarks

TFAE for HF $\mathbb{X}_1, \dots, \mathbb{X}_n$:

(a) $\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n$ f.d.a.

(b) $\chi_{\mathrm{orb}}(U_1(t)\mathbb{X}_1 U_1(t)^*, \dots, U_n(t)\mathbb{X}_1 U_n(t)^*) > -\infty$ $(\forall t > 0)$.

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  $\Rightarrow UPU^*, Q$ are in generic position.
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Moreover, all $W^*(U(t)PU(t)^*, Q)$, $t > 0$, are isomorphic.

Final Comments and questions

- Orbital approach to free entropic pressure has been developed (in preparation); it nicely fits random matrix models due to Collins, Guionnet and Segala. In fact, their models provide explicit examples whose $\chi_{\mathbf{orb}}$ and also liberation gradients are computable.

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  It is natural to ask whether or not $i^* = -\chi_{\mathrm{orb}}$ holds for those examples.
- A possible counterpart ($i^* \leq -\chi_{\mathrm{orb}}$) of Biane, Capitaine and Guionnet's result showing $\chi \leq \chi^*$ seems a big next question; probably hard.
  Remark: Dabrowski has proposed a different path to the inequality.

References

- Biane and Dabrowski, Concavification of free entropy, Adv. Math., 234 (2013), 667–696.
- Hiai, Miyamoto and U., Orbital approach to microstate free entropy, IJM, 20 (2009), 227–273.
- Izumi and U., Remarks on free mutual information and orbital free entropy, preprint, arXiv:1306.5372.
- U., Orbital free entropy, revisited, IUMJ, to appear, arXiv:1210.6421.