Overlapping Patches for Dynamic Surface Problems

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Coordinate Charts, or Patches

A collection of mappings $\mathbf{X}_i(\alpha, \beta, t) : \mathbf{R}^2 \to S$ from the plane onto the free surface.

Associated partition of unity $\{\Psi_i\}$, $\sum \Psi_i = 1$ on S.

Advantages:

- adaptivity
- complex surfaces
- isothermal coordinates

Time Evolution of Patches

Preserve partition of unity by preserving on normal lines:

$$\Psi_t = \mathbf{X}_t \cdot \nabla_s \Psi$$

Preserve physical quantities on material lines:

$$\mu_t = -\mathbf{U} \cdot \nabla_s \mu$$

Upwind considerations require:

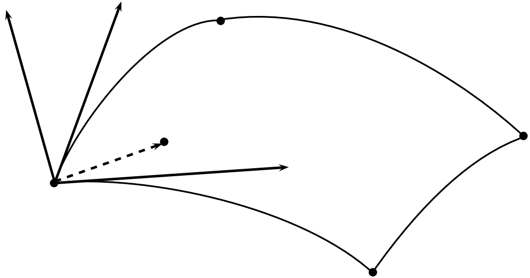
$$(\Psi\mu)_t = \mu\Psi_t + \Psi\mu_t = \mu\mathbf{X}_t \cdot \nabla_s \Psi - \Psi\mathbf{U} \cdot \nabla_s \mu$$

With reconstruction as:

$$\mu = (\sum \Psi_i)\mu = \sum(\Psi_i \mu)$$

Interpolation

At a point $\mathbf{X}^*$ on one patch, need the value of $\Psi\mu$ from other patches.



$\mathbf{X}^*$ has unknown preimage

$\alpha = \alpha_{ij} + \Delta\alpha$.

Once α is known, can easily interpolate $\Psi\mu$ there (say, bicubic).

Physical Problem

Vortex sheet (S) motion in ideal flow:

$$\begin{aligned} U_t^\pm + U^\pm \cdot \nabla U^\pm + \nabla p = 0 & \qquad \text{in } D^\pm \\ \nabla \cdot U^\pm = 0 & \qquad \text{in } D^\pm \\ \nabla \times U^\pm = 0 & \qquad \text{on } S \\ U^+ \cdot n = U^- \cdot n & \qquad \text{on } S \end{aligned}$$

Vortex sheet with strength μ induces vector potential

$$A(x) = \frac{1}{4\pi} \int_S \mu(x') n(x') \times \nabla_{x'} \left(\frac{1}{|x - x'|} \right) dS(x')$$

Physical velocity:

$$U \cdot n = (n \cdot \nabla \times A) n = \left[(A \cdot T_1)_2 - (A \cdot T_2)_1 \right] n$$

Kernel Smoothing

Regularize the fundamental solution to G_δ

$$G_\delta = -\frac{1}{4\pi}\frac{\mathrm{erf}(r/\delta)}{r} = G(x)\mathrm{erf}(r/\delta)$$

Correction Terms

$$\int - \sum\nolimits_{\delta} = \left(\int - \int_{\delta} \right) + \left(\int_{\delta} - \sum\nolimits_{\delta} \right)$$

Regularization correction:

$$\begin{aligned} \epsilon &= \int_S n(x') \times \left[\nabla_{x'} G_\delta(x - x') - \nabla_{x'} G(x - x') \right] \mu(x') dS(x') \\ &= \frac{\delta}{2\sqrt{\pi}} \left\{ T_2 \mu_1 - T_1 \mu_2 \right\} + \mathcal{O}(\delta^3) \end{aligned}$$

Discretization correction based on estimates of the Fourier coefficients of the regularized kernel, but won't fit onto a slide.

Acknowledgements and References

Joint work with M. Siegel and M. Booty (NJIT), and D. Ambrose (Drexel).

References:

- Beale: "A Grid-Based Boundary Integral Method for Elliptic Problems in Three Dimensions"
- Bruno and Kunyansky: "A Fast, High-Order Algorithm for the Solution of Surface Scattering Problems: ..."
- Caflisch and Li: "Lagrangian Theory for 3D Vortex Sheets"
- Baker and Nie: "Application of Adaptive Quadrature to Vortex Sheet Motion"