Dispersive perturbations of the Burgers equation

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Motivation

- To study the influence of dispersion on the space of resolution, on the lifespan [1], the possible blow-up and on the dynamics of solutions to the Cauchy problem for "weak" dispersive perturbations of hyperbolic quasilinear equations or systems, as for instance various models of water waves or nonlinear optics.
- Focus on the model class of equations (introduced by Whitham 1972 for a special choice of the kernel k, see below) :

$$u_t + uu_x + \int_{-\infty}^{\infty} k(x-y)u_x(y,t)dy = 0. \tag{1}$$

[1] One should not forget that most of dispersive models are not derived from first principles but as asymptotic models in various regimes, and one does not expect a priori global well-posedness

- This equation can also be written on the form

$$u_t + uu_x - Lu_x = 0, \tag{2}$$

where the Fourier multiplier operator L is defined by

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi),$$

where $p = \hat{k}$.

In the original Whitham equation, the kernel k was given by

$$k(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\tanh \xi}{\xi} \right)^{1/2} e^{ix\xi} d\xi, \tag{3}$$

that is $p(\xi) = \left(\frac{\tanh \xi}{\xi} \right)^{1/2}$.

- The dispersion is in this case that of the finite depth surface water waves without surface tension.
- With surface tension, one gets

$$p(\xi) = (1 + \beta|\xi|^2)^{1/2} \left(\frac{\tanh \xi}{\xi} \right)^{1/2}, \quad \beta \geq 0.$$

Whitham equations are also 1D version of the Full Dispersion Kadomtsev-Petviashvili (FDKP) equations introduced by D. Lannes (2013) and studied in Lannes-S (2013).

$$\partial_t u + c_{WW}(\sqrt{\mu}|D^\mu|)(1+\mu\frac{D_2^2}{D_1^2})^{1/2}u_x + \mu\frac{3}{2}uu_x = 0, \tag{4}$$

where $c_{WW}(\sqrt{\mu}k)$ is the phase velocity of the linearized water waves system, namely

$$c_{WW}(\sqrt{\mu}k) = \left(\frac{\tanh\sqrt{\mu}k}{\sqrt{\mu}k}\right)^{1/2}$$

and

$$|D^\mu| = \sqrt{D_1^2 + \mu D_2^2}, \quad D_1 = \frac{1}{i}\partial_x, \quad D_2 = \frac{1}{i}\partial_y.$$

Denoting by h a typical depth of the fluid layer, a a typical amplitude of the wave, λ_x and λ_y typical wave lengths in x and y respectively, the relevant regime here is when

$$\mu \sim \frac{a}{h} \sim \left(\frac{\lambda_x}{\lambda_y}\right)^2 \sim \left(\frac{h}{\lambda_x}\right)^2 \ll 1.$$

When adding surface tension effects, one has to replace (4) by

$$\partial_t u + \tilde{c}_{WW}(\sqrt{\mu}|D^\mu|)(1+\mu\frac{D_2^2}{D_1^2})^{1/2}u_x + \mu\frac{3}{2}uu_x = 0, \qquad (5)$$

with

$$\tilde{c}_{WW}(\sqrt{\mu}k) = (1+\beta\mu k^2)^{\frac{1}{2}}\left(\frac{\tanh\sqrt{\mu}k}{\sqrt{\mu}k}\right)^{1/2},$$

where $\beta > 0$ is a dimensionless coefficient measuring the surface tension effects,

$$\beta = \frac{\sigma}{\rho g h^2},$$

where σ is the surface tension coefficient ($\sigma = 7.10^{-3} N \cdot m^{-1}$ for the air-water interface), g the acceleration of gravity, and ρ the density of the fluid.

The general idea is to investigate the "fight" between nonlinearity and dispersion. Usually this problem is attacked by fixing the dispersion (*eg* that of the KdV equation) and varying the nonlinearity (say $u^p u_x$ in the context of generalized KdV).

Our viewpoint, which is probably more physically relevant, is to fix the quadratic nonlinearity (*eg* uu_x) and to vary (lower) the dispersion. In fact in many problems arising from Physics or Continuum Mechanics the nonlinearity is quadratic, with terms like $(u \cdot \nabla)u$ and the dispersion is in some sense weak. In particular the dispersion is not strong enough for yielding the dispersive estimates that allows to solve the Cauchy problem in relatively large functional classes (like the KdV or Benjamin-Ono equation in particular), down to the energy level for instance.[2]

[2] And thus obtaining *global well-posedness* from the conservation laws.

Many physically sounded dispersive systems have the form

$$\partial_t U + \mathcal{B}U + \epsilon\, \mathcal{A}(U, \nabla U) + \epsilon \mathcal{L} U = 0, \tag{6}$$

where the order 0 part $\partial_t U + \mathcal{B}U$ is linear hyperbolic, $\mathcal{L}$ being a linear (not necessarily skew-adjoint) dispersive operator and $\epsilon > 0$ is a small parameter which measures the (comparable) nonlinear and dispersive effects. Both the linear part and the dispersive part may involves nonlocal terms.

Boussinesq systems for surface water waves are important examples of somewhat similar systems. Note however that the Boussinesq systems (7) cannot be reduced exactly to the form (20) except when $b = c = 0$. Otherwise the presence of a "BBM like " term induces a smoothing effect on one or both nonlinear terms. They write

$$\begin{cases} \partial_t \eta + \operatorname{div} \mathbf{v} + \epsilon \operatorname{div}(\eta \mathbf{v}) + \epsilon(a \operatorname{div} \Delta \mathbf{v} - b \Delta \eta_t) = 0 \\ \partial_t \mathbf{v} + \nabla \eta + \epsilon \frac{1}{2} \nabla(|\mathbf{v}|^2) + \epsilon(c \nabla \Delta \eta - d \Delta \mathbf{v}_t) = 0 \end{cases}, \quad (x_1, x_2) \in \mathbb{R}^2, \ t \in \mathbb{R}. \tag{7}$$

where a, b, c, d are modelling constants satisfying the constraint $a + b + c + d = \frac{1}{3}$ and ad hoc conditions implying that the well-posedness of linearized system at the trivial solution $(0, \mathbf{0})$. When $b > 0, d > 0$ are not zero, the dispersion in (7) is "weak" (the corresponding linear operator is of order $-1, 0$ or 1 contrary to the case $b = d = 0, \ a < 0, c < 0$ when it is of order 3 as in the KdV equation.

Restrict to the toy model (fKdV)

$$\partial_t u - D^\alpha \partial_x u + u\partial_x u = 0, \quad (8)$$

where $x,\ t \in \mathbb{R}$, $\widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$.

- $\alpha = 1$: Benjamin-Ono. $\alpha = 2$: KdV.
- Extensively studied for $1 \leq \alpha \leq 2$ (Fonseca-Linares-Ponce, 2012-2013) : GWP.
- $\alpha = -1$: Burgers-Hilbert.
- $\alpha = -\frac{1}{2}$, reminiscent of the orignal Whitham equation.
- We focus here on the case $0 < \alpha < 1$. As previously observed $\alpha = \frac{1}{2}$ is somewhat reminiscent of the linear dispersion of finite depth water waves with surface tension.

The following quantities are conserved by the flow associated to (8),

$$M(u) = \int_{\mathbb{R}} u^2(x,t)dx, \tag{9}$$

and the Hamiltonian

$$H(u) = \int_{\mathbb{R}} \Big(\frac{1}{2}|D^{\frac{\alpha}{2}} u(x,t)|^2 - \frac{1}{6}u^3(x,t)\Big)dx. \tag{10}$$

By Sobolev $H^{\frac{1}{6}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$, and $H(u)$ is well-defined when $\alpha \geq \frac{1}{3}$. Moreover, equation (8) is invariant under the scaling transformation

$$u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1} t), \quad \forall \lambda > 0.$$

Straightforward computation : $\|u_\lambda\|_{\dot{H}^s} = \lambda^{s+\alpha-\frac{1}{2}} \|u_\lambda\|_{\dot{H}^s}$, and thus the critical index corresponding to (8) is $s_\alpha = \frac{1}{2} - \alpha$. In particular, equation (8) is L^2-critical for $\alpha = \frac{1}{2}$.

- The case $0 < \alpha < \frac{1}{3}$ is energy supercritical.
- For the GKDV equations

$$u_t + u^p u_x + u_{xxx} = 0,$$

the L^2 critical case corresponds to $p = 4$. There is no energy critical case.

- Theoretical results on the blow-up of the L^2 critical gKdV equation : Martel-Merle (2002), Martel-Merle-Raphaël (2012).
- Numerical simulations of the L^2- supercritical gKdV : Bona-Dougalis-Karakashian-McKinney (1995), Klein-Peter (2013).
- Numerical simulations of the L^2- critical gKdV : Klein-Peter (2013).

Basic questions

- How the space of resolution of the Cauchy problem is enhanced when $0 < \alpha < 1$?
- Blow-up and what kind of blow-up ?
- Solitary waves.
- Structure of the solution when it is global (decomposition into solitary waves + dispersion ?).
- Lifespan of the solutions when a small parameter ϵ is introduced.

Easy results

By using standard compactness methods, one can prove that the Cauchy problem associated to (8) is locally well-posed in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$.
Moreover, interpolation arguments or the following Gagliardo-Nirenberg inequality,

$$\|u\|_{L^3} \lesssim \|u\|_{L^2}^{\frac{3\alpha-1}{3\alpha}} \|D^{\frac{\alpha}{2}} u\|_{L^2}^{\frac{1}{3\alpha}}, \quad \alpha \geq \frac{1}{3},$$

combined with the conserved quantities M and H defined in (9) and (10) implies the existence of global weak solution in the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ as soon as $\alpha > \frac{1}{2}$ and for small data in $H^{\frac{1}{4}}(\mathbb{R})$ when $\alpha = \frac{1}{2}$. More precisely[3] :

Theorem

Let $\frac{1}{2} < \alpha < 1$ and $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$. Then (8) possesses a global weak solution in $L^\infty([0,T]; H^{\frac{\alpha}{2}}(\mathbb{R}))$ with initial data u_0. The same result holds when $\alpha = \frac{1}{2}$ provided $\|u_0\|_{L^2}$ is small enough.

[3]We recall that we exclude the value $\alpha = 1$ which corresponds to the Benjamin-Ono equation for which much more complete results are known.

Moreover, it was established by Ginibre and Velo (1991) that a Kato type local smoothing property holds, implying global existence of weak L^2 solutions :

Theorem

Let $\frac{1}{2} < \alpha < 1$ and $u_0 \in L^2(\mathbb{R})$. Then (8) *possesses a global weak solution in $L^\infty([0,\infty); L^2(\mathbb{R})) \cap l^\infty L^2_{loc}(\mathbb{R}; H^{\frac{\alpha}{2}}_{loc}(\mathbb{R}))$ with initial data u_0.*

- However, the case $0 < \alpha < \frac{1}{2}$ is more delicate and the previous results are not known to hold. In particular the Hamiltonian H together with the L^2 norm do not control the $H^{\frac{\alpha}{2}}(\mathbb{R})$ norm anymore. Note that the Hamiltonian does not make sense when $0 < \alpha < \frac{1}{3}$ (energy supercritical).

The local theory (F. Linares-D. Pilod-JCS SIMA 2014)

Theorem

Let $0 < \alpha < 1$. *Define* $s(\alpha) = \frac{3}{2} - \frac{3\alpha}{8}$ *and assume that* $s > s(\alpha)$. *Then, for every* $u_0 \in H^s(\mathbb{R})$, *there exists a positive time* $T = T(\|u_0\|_{H^s})$ *(which can be chosen as a nonincreasing function of its argument), and a unique solution* u *to* (8) *satisfying* $u(\cdot, 0) = u_0$ *such that*

$$u \in C([0,T] : H^s(\mathbb{R})) \quad \text{and} \quad \partial_x u \in L^1([0,T] : L^\infty(\mathbb{R})). \tag{11}$$

Moreover, for any $0 < T' < T$, *there exists a neighborhood* $\mathcal{U}$ *of* u_0 *in* $H^s(\mathbb{R})$ *such that the flow map data-solution*

$$S^s_{T'} : \mathcal{U} \longrightarrow C([0,T']; H^s(\mathbb{R})), \ u_0 \longmapsto u, \tag{12}$$

is continuous.

Remarks

- It is a classical result that the IVP associated to the Burgers equation is ill-posed in $H^{\frac{3}{2}}(\mathbb{R})$.
- When $\alpha = 1$, the exponent $s(\alpha)$ corresponds to $\frac{9}{8}$ obtained for the BO equation in Kenig-Koenig (2003). The index $s(\alpha)$ is probably not optimal.
- It has been proven in Molinet-S-Tzvetkov (2001) that, for $0 < \alpha < 2$ the Cauchy problem is C^2- ill-posed[4] for initial data in any Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, and in particular that the Cauchy problem cannot be solved by a Picard iterative scheme implemented on the Duhamel formulation.
- The problem to prove well-posedness in $H^{\frac{\alpha}{2}}(\mathbb{R})$ in the case $\frac{1}{2} \leq \alpha < 1$, which would imply global well-posedness by using the conserved quantities (9) and (10), is still open. This conjecture is supported by the numerical simulations in C. Klein-S (see below) that suggest that the solution is global in this case, for arbitrary large initial data.

[4] That is that the flow map cannot be C^2.

- Theorem 3 extends easily by perturbation to some non pure power dispersions. For instance, in the case of the Whitham equation with surface tension, it suffices to observe that

$$(1+\xi^2)^{1/2}\left(\frac{\tanh|\xi|}{|\xi|}\right)^{1/2} = |\xi|^{1/2} + R(|\xi|),$$

where $|R(|\xi|)| \leq |\xi|^{-3/2}$ for large $|\xi|$.

Ideas on the proof

- Since we cannot prove Theorem 3 by a contraction method as explained above, we use a compactness argument. Standard energy estimates, the Kato-Ponce commutator estimate and Gronwall's inequality provide the following bound for smooth solutions

$$\|u\|_{L^\infty_T H^s_x} \leq c\|u_0\|_{H^s_x} e^{c\int_0^T \|\partial_x u\|_{L^\infty_x} dt}.$$

Therefore, it is enough to control $\|\partial_x u\|_{L^1_T L^\infty_x}$ at the H^s-level to obtain our *a priori* estimates.

- Note that the classical Strichartz estimate for the free group $e^{tD^\alpha \partial_x}$ associated to the linear part of (8), and derived by Kenig, Ponce and Vega (1991), induces a loss of $\frac{1-\alpha}{4}$ derivatives in L^∞, since we are in the case $0 < \alpha < 1$. Then, we need to use a refined version of this Strichartz estimate, derived by chopping the time interval in small pieces whose length depends on the spatial frequency of the function (see below). This estimate was first established by Kenig and Koenig (2003) (based on previous ideas of Koch and Tzvetkov) in the Benjamin-Ono context (when $\alpha = 1$) .
- We also use a maximal function estimate for $e^{tD^\alpha \partial_x}$ in the case $0 < \alpha < 1$, which follows directly from the arguments of Kenig, Ponce and Vega (1991).
- To complete our argument, we need a local smoothing effect for the solutions of the nonlinear equation (8), which is based on series expansions and remainder estimates for commutator of the type $[D^\alpha \partial_x, u]$ derived by Ginibre and Velo (1989).
- All those estimates allow us to obtain the desired *a priori* bound for $\|\partial_x u\|_{L^1_T L^\infty_x}$ at the H^s-level, when $s > s(\alpha) = \frac{3}{2} - \frac{3\alpha}{8}$, *via* a recursive argument. Finally, we conclude the proof of Theorem 3, by applying the same method to the differences of two solutions of (8) and by using the so-called Bona-Smith argument.

The refined Strichartz estimate for or solutions of the nonhomogeneous linear equation

$$\partial_t u - D^\alpha \partial_x u = F \; . \tag{13}$$

Proposition

Assume that $0 < \alpha < 1$, $T > 0$ and $\delta \geq 0$. Let u be a smooth solution to (13) defined on the time interval $[0, T]$. Then, there exist $0 < \kappa_1, \; \kappa_2 < \frac{1}{2}$ such that

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim T^{\kappa_1} \|J^{1+\frac{\delta}{4}+\frac{1-\alpha}{4}+\theta} u\|_{L^\infty_T L^2_x} + T^{\kappa_2} \|J^{1-\frac{3\delta}{4}+\frac{1-\alpha}{4}+\theta} F\|_{L^2_{T,x}} \; , \tag{14}$$

for any $\theta > 0$.

Remark

In our analysis, the optimal choice in the estimate above corresponds to $\delta = 1 - \frac{\alpha}{2}$. Indeed, if we denote $a = 1 + \frac{\delta}{4} + \frac{1-\alpha}{4} + \theta$ and $b = 1 - \frac{3\delta}{4} + \frac{1-\alpha}{4} + \theta$, we should adapt δ to get $a = b + 1 - \frac{\alpha}{2}$, since we need to absorb 1 derivative appearing in the nonlinear part of (8) and we are able to recover $\frac{\alpha}{2}$ derivatives by using the smoothing effect associated with solutions of (13). The use of $\delta = 1 - \frac{\alpha}{2}$ in estimate (14) provides the optimal regularity $s > s(\alpha) = \frac{3}{2} - \frac{3\alpha}{8}$ in Theorem 3.

An ill-posedness result

As in Birnir et al (1996) for the GKdV and NLS equations, one can use the solitary wave solutions to disprove the uniform continuity of the flow map for the Cauchy problem under suitable conditions. More precisely, we consider again the initial value problem (IVP)

$$\begin{cases} \partial_t u - D^\alpha \partial_x u + u\,\partial_x u = 0, \quad x,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{15}$$

Proposition

If $1/3 \leq \alpha \leq 1/2$, then the IVP (15) is ill-posed in $H^{s_\alpha}(\mathbb{R})$ with $s_\alpha = \frac{1}{2} - \alpha$, in the sense that the time of existence T and the continuous dependence cannot be expressed in terms of the size of the data in the H^{s_α}-norm. More precisely, there exists $c_0 > 0$ such that for any δ, $t > 0$ small there exist data u_1, $u_2 \in \mathcal{S}(\mathbb{R})$ such that

$$\|u_1\|_{s,2} + \|u_2\|_{s,2} \leq c_0, \quad \|u_1 - u_2\|_{s,2} \leq \delta, \quad \|u_1(t) - u_2(t)\|_{s,2} > c_0,$$

where $u_j(\cdot)$ denotes the solution of the IVP (15) with data u_j, $j = 1, 2$.

Solitary waves

A (localized) solitary wave solution of (8) of the form $u(x,t) = Q_c(x - ct)$ must satisfy the equation

$$D^\alpha Q_c + cQ_c - \frac{1}{2} Q_c^2 = 0, \tag{16}$$

where $c > 0$.

One does not expect solitary waves to exist when $\alpha < \frac{1}{3}$ since then the Hamiltonian does not make sense (see a formal argument in Kuznetsov-Zakharov 2000). In fact :

Theorem

Assume that $0 < \alpha \le \frac{1}{3}$. *Then* (16) *does not possesses any nontrivial solution* Q_c *in the class* $H^{\frac{\alpha}{2}}(\mathbb{R}) \cap L^3(\mathbb{R})$[5]. *(The proof works as well for* $\alpha < 0$*).*

Based on the identity

$$\int_{\mathbb{R}} (D^\alpha \phi) x \phi' dx = \frac{\alpha - 1}{2} \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} \phi|^2 dx,$$

[5] This implies that the Hamiltonian is well defined.

The solitary waves are obtained following Weinstein classical approach by looking for the best constant $C_{p,\alpha}$in the Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}} |u|^{p+2} \leq C_{p,\alpha} \left(\int_{\mathbb{R}} |D^{\alpha/2} u|^2 \right)^{\frac{p}{2\alpha}} \left(\int_{\mathbb{R}} |u|^2 \right)^{\frac{p}{2\alpha}(\alpha-1)+1}, \quad \alpha \geq \frac{p}{p+2}. \tag{17}$$

This amounts to minimize the functional

$$J^{p,\alpha}(u) = \frac{\left(\int_{\mathbb{R}} |D^{\alpha/2} u|^2\right)^{\frac{p}{2\alpha}} \left(\int_{\mathbb{R}} |u|^2\right)^{\frac{p}{2\alpha}(\alpha-1)+1}}{\int_{\mathbb{R}} |u|^{p+2}}. \tag{18}$$

In our setting, that is with $p = 1$ and one obtains (see Frank-Lenzman 2010 and the references therein) :

Theorem

Let $\frac{1}{3} < \alpha < 1$. Then

(i) Existence : There exists a solution $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ of equation (16) such that $Q = Q(|x|) > 0$ is even, positive and strictly decreasing in $|x|$. Moreover, the function $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ is a minimizer for $J^{p,\alpha}$.

(ii) Symmetry and Monotonicity : If $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ is a nontrivial solution of (16) with $Q \geq 0$, then there exists $x_0 \in \mathbb{R}$ such that $Q(\cdot - 0)$ is an even, positive and strictly decreasing in $|x - x_0|$.

(iii) Regularity and Decay : If $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ solves (16), then $Q \in H^{\alpha+1}(\mathbb{R})$. Moreover, we have the decay estimate $|Q(x)| + |xQ'(x)| \leq \frac{C}{1+|x|^{1+\alpha}}$, for all $x \in \mathbb{R}$ and some constant $C > 0$.

Uniqueness issues have been addressed in Frank-Lenzman 2010 (in any dimension). They concern ground states solutions according to the following definition

Definition

Let $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ be an even and positive solution of (16) . If

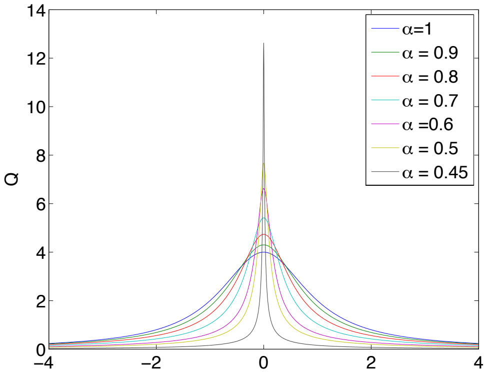
$$J^{(p,\alpha)}(Q) = \inf \left\{ J^{(p,\alpha)}(u) \ : \ u \in H^{\frac{\alpha}{2}}(\mathbb{R}) \setminus \{0\} \right\},$$

then we say that Q is a ground state solution.

The main result in Frank-Lenzman 2010 implies in our case ($p = 1$) that the ground state is unique when $\alpha > \frac{1}{3}$.
Observe that the uniqueness (up to the trivial symmetries) of the solitary-waves of the Benjamin-Ono solutions has been established by Amick-Toland (1991).
Note that the method of proof of the existence Theorem does not yields any (orbital) stability result. One has to use instead a variant of the Cazenave-Lions method, that is obtain the solitary waves by minimizing the Hamiltonian with fixed L^2 norm. This has been done in Albert-Bona-S (1997) in the case $\alpha = 1$

- Ehrnström, Groves and Wahlen (Nonlinearity 2012) have shown that the original Whitham equation possesses solitary waves. A crucial point in the proof is that the Whitham equation "reduces" to the KdV equation in the long wave limit.
- See also Ehrnström-Kalish (2009) for periodic traveling waves and Hur-Johnson (2013) for their stability properties.

Numerical simulations : solitary waves for $c = 1$, different α (*Klein & Saut '14*)



Long time existence issues (with respect to the inverse of a small parameter)

Concerning the Burgers -Hilbert equation

$$\partial_t u + \mathcal{H}u + \epsilon u \partial_x u = 0, \tag{19}$$

where $\mathcal{H}$ is the Hilbert transform,
J.K. Hunter and M.Ifrim have obtained the rather unexpected result (SIMA 2012) (see also another proof in Hunter-Ifrim-Tataru-Wang 2013) :

Theorem

Suppose that $u_0 \in H^2(\mathbb{R})$. There are constants $k > 0$ and $\epsilon_0 > 0$, depending only on $|u_0|_{H^2}$, such that for every ϵ with $|\epsilon| \leq \epsilon_0$, there exists a solution $u \in C(I_\epsilon; H^2(\mathbb{R}) \cap C^1(I_\epsilon; H^1(\mathbb{R}))$ of BH defined on the time-interval $I_\epsilon = [-k/\epsilon^2, k/\epsilon^2]$.

So the existence time is enhanced thanks to the order zero operator $\mathcal{H}$.

- Likely to work (with a different lifespan ?) for the equation

$$u_t + \epsilon u u_x - D^\alpha u_x = 0, \quad -1 < \alpha < 0.$$

Back to

$$\partial_t U + \mathcal{B}U + \epsilon\, \mathcal{A}(U, \nabla U) + \epsilon \mathcal{L}U = 0, \tag{20}$$

where the 0^{th} order part $\partial_t U + \mathcal{B}U$ is linear hyperbolic, $\mathcal{L}$ being a linear (not necessarily skew-adjoint) dispersive operator and $\epsilon > 0$ is a small parameter which measures the (comparable) nonlinear and dispersive effects. Both the linear part and the dispersive part may involves nonlocal terms

- Basic question : is the hyperbolic lifespan $1/\epsilon$ enhanced by the dispersion.
- Trivial in the scalar 1D case where $\mathcal{B}U = u_x$ can be eliminated, leading to

$$u_t + \epsilon f(u)_x - \epsilon L u_x = 0, \tag{21}$$

for which existence on time scales of order $1/\epsilon$ is trivial.

Actually, whatever the dispersive term L, one has the dichotomy : either the solution is global, either its life span has order $0(1/\epsilon)$, as immediately seen by the change of the time variable $\tau = t/\epsilon$ which reduces (21) to

$$u_\tau + f(u)_x - L u_x = 0, \tag{22}$$

$$
(B_{abcd}) \begin{cases} \zeta_t + \nabla\cdot\mathbf{v} + \epsilon[\nabla\cdot(\zeta\mathbf{v}) + a\nabla\cdot\Delta\mathbf{v} - b\Delta\zeta_t] = 0 \\ \mathbf{v}_t + \nabla\zeta + \epsilon[\frac{1}{2}\nabla|\mathbf{v}|^2 + c\nabla\Delta\zeta - d\Delta\mathbf{v}_t] = 0. \end{cases}
$$

For the Boussinesq systems, things are not easy, even to obtain the hyperbolic lifespan $1/\epsilon$ (see Li Xu-JCS 2012 for most of the Boussinesq systems by symmetrization techniques).

- The proofs using mainly dispersion (that is high frequencies) do not take into account the algebra (structure) of the nonlinear terms. They allows initial data in relatively big Sobolev spaces but seem to give only existence times of order $O(1/\sqrt{\epsilon})$, eg in Linares-Pilod-S. in the "KdV-KdV" 2D case.
- The existence proofs on existence times of order $1/\epsilon$ are of "hyperbolic" nature. They do not take into account the dispersive effects (treated as perturbations).
- Is it possible to go till $O(1/\epsilon^2)$, or to get global existence. Plausible in one D (the Boussinesq systems should evolves into an uncoupled system of KdV equations). Not so clear in 2D... One should there use dispersion. Use of a normal form technique (*à la Germain-Masmoudi-Shatah*) ? Possible difficulties due to the dispersion relation.
- For the fKdV equation, numerics (see below) suggest global existence for small data.

The fBBM equation :

$$\partial_t u + \partial_x u + u\partial_x u + D^\alpha \partial_t u = 0, \tag{23}$$

The case $\alpha = 2$ corresponds to the classical BBM equation, $\alpha = 1$ to the BBM version of the Benjamin-Ono equation.
For any α the energy

$$E(t) = \int_{\mathbb{R}} (u^2 + |D^{\frac{\alpha}{2}} u|^2) dx$$

is formally conserved. By a standard compactness method this implies that the Cauchy problem for (23) admits a global weak solution in $L^\infty(\mathbb{R}; H^{\frac{\alpha}{2}}(\mathbb{R}))$ for any initial data $u_0 = u(\cdot, 0)$ in $H^{\frac{\alpha}{2}}(\mathbb{R})$.
One can also use the equivalent form

$$\partial_t u + \partial_x (I + D^\alpha)^{-1} \left(u + \frac{u^2}{2} \right) = 0, \tag{24}$$

which gives the Hamiltonian formulation

$$u_t + J_\alpha \nabla_u H(u) = 0$$

where the skew-adjoint operator J_α is given by $J_\alpha = \partial_x(I + D^\alpha)^{-1}$ and $H(u) = \frac{1}{2}\int_{\mathbb{R}}(u^2 + \frac{1}{3}u^3)$. Note that the Hamiltonian makes for $u \in H^{\frac{\alpha}{3}}(\mathbb{R})$ if and only if $\alpha \geq \frac{1}{3}$.
The form (24) shows clearly that the fractionary BBM equation is for $0 < \alpha < 1$ a kind of "dispersive regularization" of the Burgers equation.

Theorem

(Linares-Pilod-S. 2013)
Let $0 < \alpha < 1$. *Then the Cauchy problem for* (23) *or* (24) *is locally well-posed for initial data in* $H^r(\mathbb{R})$, $r > r_\alpha = \frac{3}{2} - \alpha$.

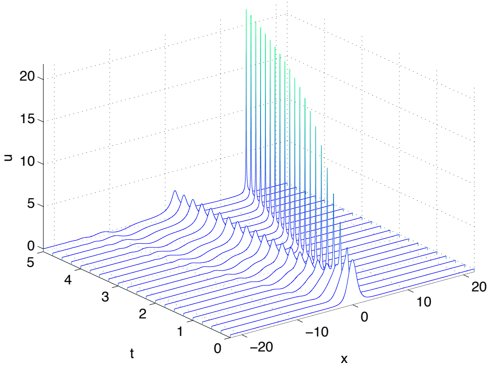
- Proof based on energy estimates.
- It would be interesting to lower the value of r_α, in particular down to the energy level $r = \frac{\alpha}{2}$, or to prove an ill-posedness result for $r < r_\alpha$.

Blow-up issues

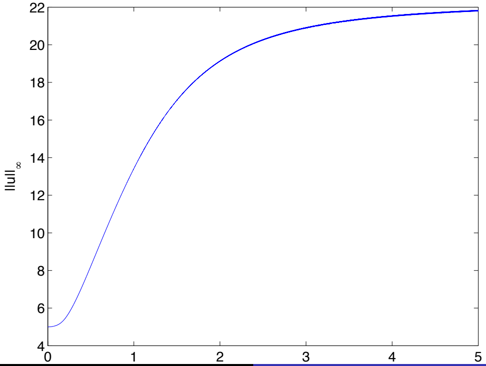
- The question is when $0 < \alpha < 1$, to look for a possible blow-up of the local solution and to to examine its nature. Kuznetsov-Zakharov (2000) claim (without proof) that there is no shock formation.
- When $-1 \leq \alpha < 0$, the question of blow-up is positively answered by Castro-Córdoba-Gancedo (2010) (the proof can be easily extended to the Whitham equation). See also Naumkin-Shishmarev (1994), Constantin-Escher (1998) for related equations. The proofs (by contradiction) do not give information on the type of blow-up.
- No rigorous results when $0 < \alpha < 1$, but strong numerical evidences (see below).

Numerical results for the fKdV, the fBBM and the Whitham equation (Christian Klein-JCS 2014).

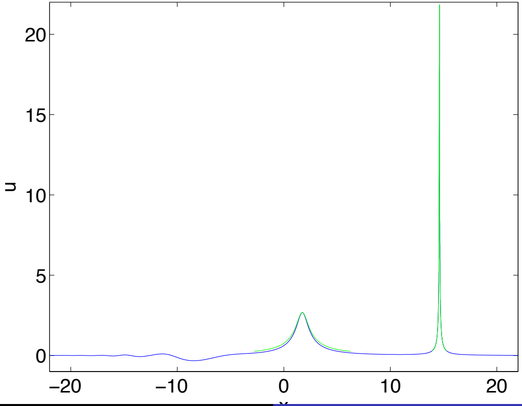
L^2-subcritical case $\alpha = 0.6$. $u_0 = 5\text{sech}^2 x$



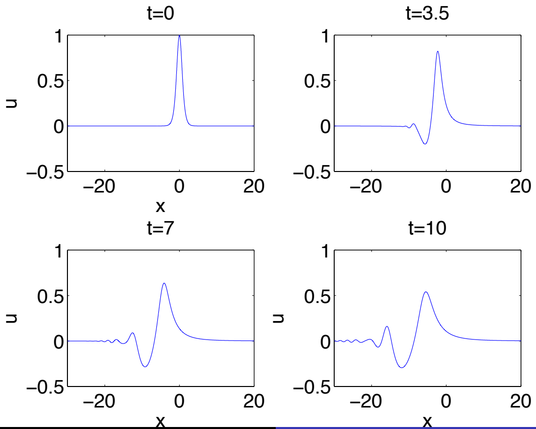
L^2-subcritical case $\alpha = 0.6$. $u_0 = 5\text{sech}^2 x$. Evolution of the sup norm



$\alpha = 0.6$. $u_0 = 5\text{sech}^2 x$. Fitted soliton at humps in green



L^2 critical case $\alpha = 0.5$. $u_0 = \mathrm{sech}^2 x$.



L^2-critical case $\alpha = \frac{1}{2}$. $u_0 = 3\text{sech}^2 x$.

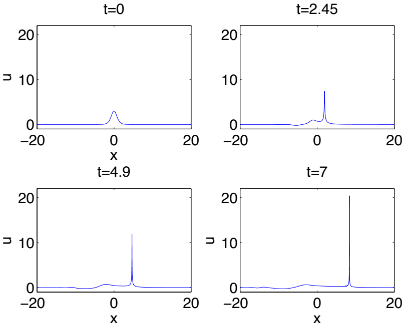
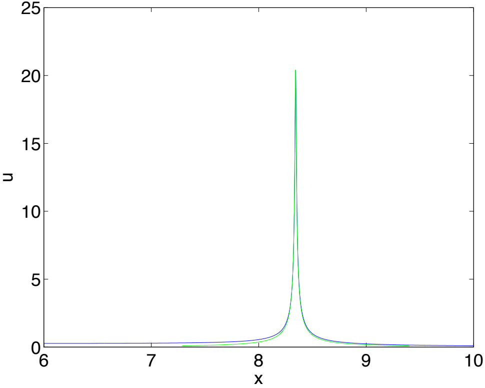
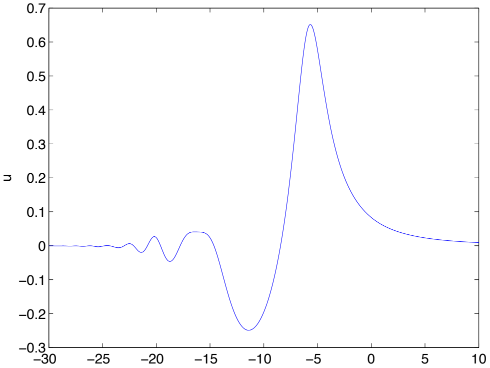


FIGURE 4. Solution to the fKdV equation (7) for $\alpha = 0.5$ and the initial data $u_0 = 3\text{sech}^2 x$ for several values of t.

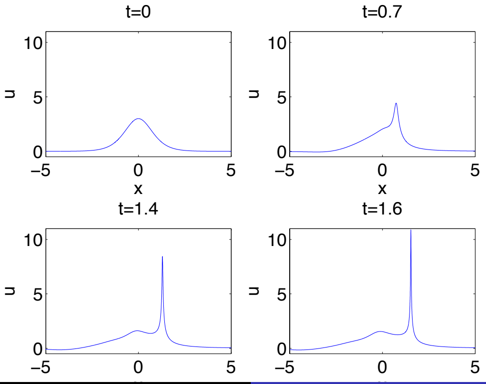
$\alpha = \frac{1}{2}$. $u_0 = 3\text{sech}^2 x$. Fit with rescaled soliton (green)



L^2-supercritical & Energy subcritical $\alpha = 0.45$. $u_0 = \text{sech}^2 x$. $t = 10$



L^2-supercritical & Energy subcritical $\alpha = 0.45$. $u_0 = 3\text{sech}^2 x$



L^2-supercritical & Energy subcritical $\frac{1}{3} < \alpha = 0.45$ $u_0 = \text{sech}^2 x$

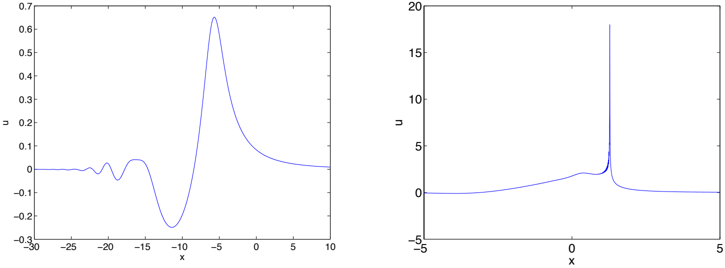


FIGURE 10. Solution to the fKdV equation (7) for $\alpha = 0.4$, on the left for the initial data $u_0 = \text{sech}^2 x$ at $t = 10$, on the right for the initial data $u_0 = 3\text{sech}^2 x$ at $t = 1.11$.

Energy supercritical $0 < \alpha < \frac{1}{3}. u_0 = \operatorname{sech}^2 x$

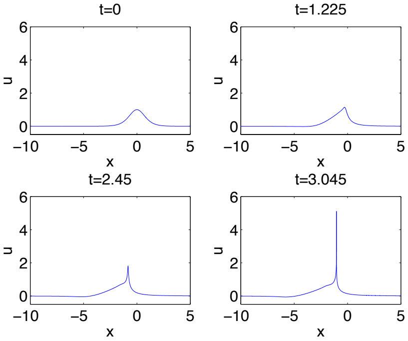
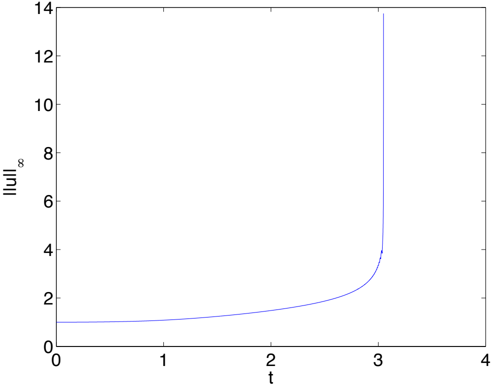
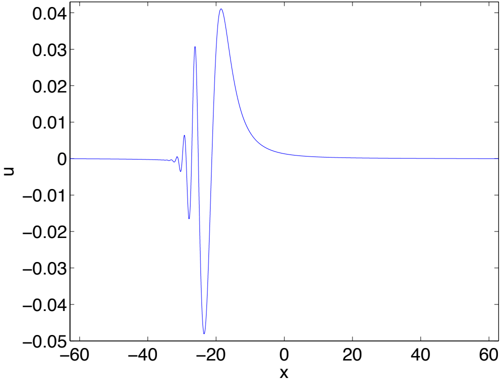


FIGURE 7. Solution to the fKdV equation (7) for $\alpha = 0.2$ and the initial data $u_0 = \operatorname{sech}^2 x$ for several values of t.

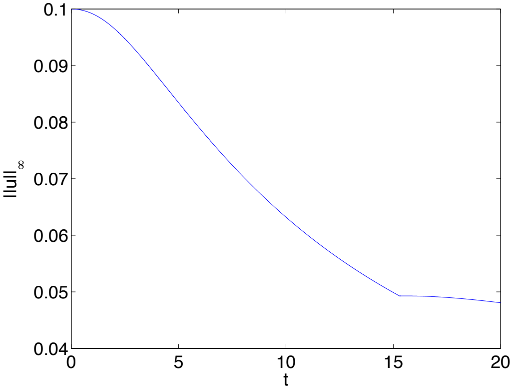
Energy supercritical $\alpha = 0.2. u_0 = \mathrm{sech}^2 x.$ $||u||_\infty$



Energy supercritical $\alpha = 0.2. u_0 = 0.1\text{sech}^2 x, t = 20$



Energy supercritical $\alpha = 0.2. u_0 = 0.1\text{sech}^2 x, ||u||_\infty$



Conjectures for the fKdV equation

Let $u_0 \in L_2(\mathbb{R})$ smooth and with a single hump. Then for

- $\alpha > 0.5$: solutions to the fKdV equations with the initial data u_0 stay smooth for all t. For large t they decompose asymptotically into solitons and radiation.
- $0 < \alpha \leq 0.5$: solutions to the fKdV equations with initial data u_0 sufficiently small, but non-zero mass stay smooth for all t.
- $\alpha = 0.5$: solutions to the fKdV equations with the initial data u_0 with negative energy and mass larger than the soliton mass blow up at finite time t^* and infinite x^*. The type of the blow-up for $t \nearrow t^*$ is characterized by

$$u(x,t) \sim \frac{1}{\sqrt{L(t)}} Q_1\left(\frac{x - x_m}{L(t)}\right), \quad L = c_0(t^* - t), \tag{25}$$

where c_0 is a constant, and where Q_1 is the solitary wave solution (16) for $c = 1$. In addition one has

$$||u_x||_2 \sim \frac{1}{L^2(t)}. \tag{26}$$

- $1/3 < \alpha < 0.5$: solutions to the fKdV equations with the initial data u_0 and sufficiently large L_2 norm blow up at finite time t^* and finite $x = x^*$. A soliton-type hump separates from the initial hump and eventually blows up. The type of the blow-up for $t \nearrow t^*$ is characterized by

$$u(x,t) \sim \frac{1}{L^\alpha(t)} U\left(\frac{x - x_m}{L(t)}\right), \quad L = c_1(t^* - t)^{\frac{1}{1+\alpha}}, \tag{27}$$

where c_1 is a constant, and where U is a solution of equation

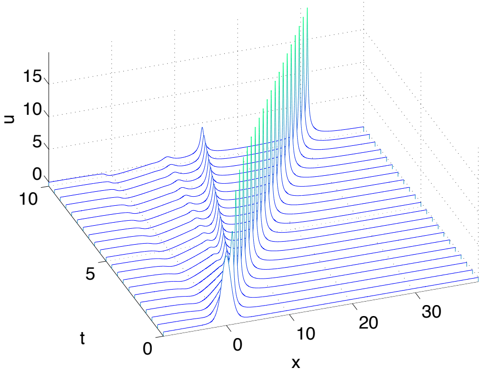
$$-a_\infty\left(\alpha U^\infty + yU^\infty_y\right) - v_\infty U^\infty_y + U^\infty U^\infty_y - D^\alpha_y U^\infty_y = 0. \tag{28}$$

vanishing for $|y| \to \infty$ (if such a solution exists). In addition one has

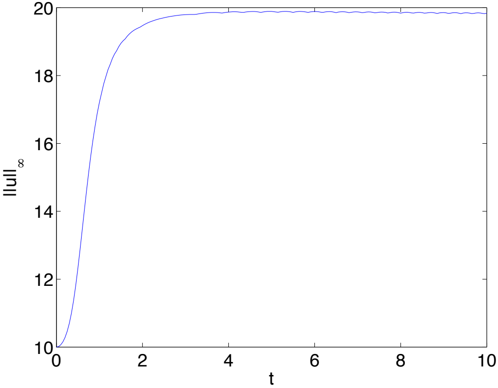
$$||u_x||_2 \sim \frac{1}{L^{2\alpha+1}(t)}. \tag{29}$$

- $0 < \alpha < 1/3$: solutions to the fKdV equations with the initial data u_0 and sufficiently large L_2 norm blow up at finite time t^* and finite $x = x^*$. The nature of blow-up is different from the previous one since no solitary waves exist in this case, the maximum of the initial hump evolves directly into a blow-up. Thus the blow-up seems to be different from that occurring in the supercritical gKdV equation when $p > 4$. But the blow-up profile appears to be still given by (27).

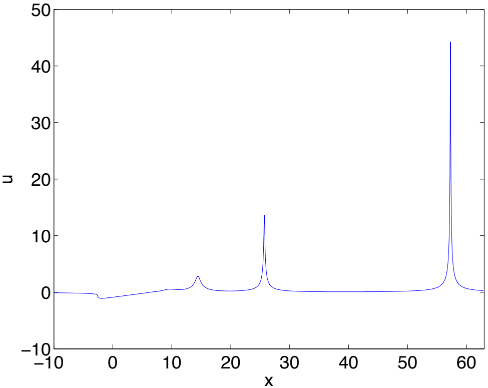
fBBM $\alpha = 0.5. u_0 = 10\text{sech}^2 x$



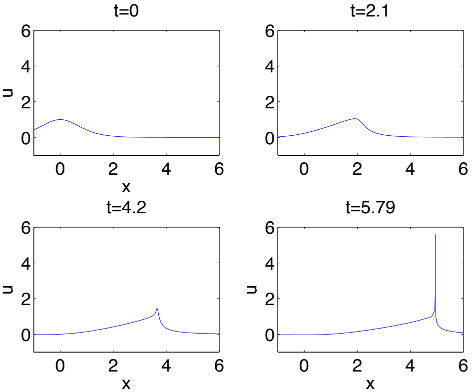
fBBM $\alpha = 0.5. u_0 = 10\text{sech}^2 x$



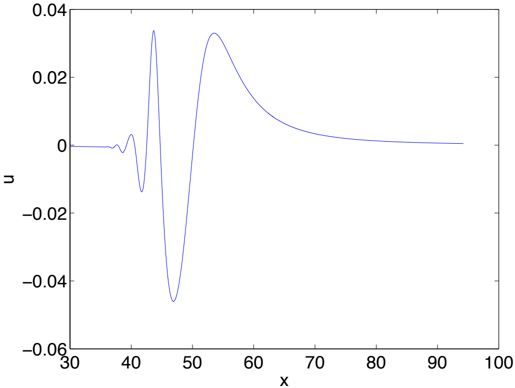
fBBM $\alpha = 0.5. u_0 = 20\text{sech}^2 x,\ t = 10$



fBBM $\alpha = 0.2$, $u_0 = \text{sech}^2 x$



fBBM $\alpha = 0.2$, $u_0 = 0.1\text{sech}^2 x$, $t = 100$



Conjectures for the fBBM equation

Consider smooth initial data $u_0 \in L_2(\mathbb{R})$ with a single hump. Then for

- $\alpha > 1/3$: solutions to the fBBM equations with the initial data u_0 stay smooth for all t. For large t they decompose asymptotically into solitons and radiation.
- $0 < \alpha \leq 1/3$: solutions to the fKdV equations with the initial data u_0 and sufficiently large L_2 norm form a cusp of the form $|x - x^*|^\alpha$ at finite time t^* and finite $x = x^*$.

  Solutions with sufficiently small initial data are global.
- The fBBM solitons (16) are stable for $\alpha > 1/3$.

Remark

We note here a strong contrast with the gKdV and the generalized BBM equation

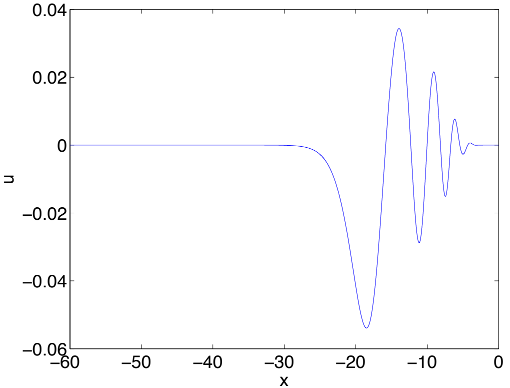
$$u_t + u_x + u^p u_x - u_{xxt} = 0. \tag{30}$$

For both the gKdV and (30), the critical exponent for the stability of solitary waves is $p = 4$, though the explanation for instability when $p \geq 4$ is different since no blow-up occurs for (30), whatever p.

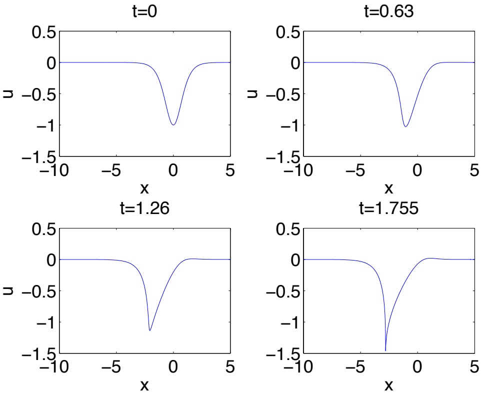
For the fKdV and fBBM equations the critical exponents seem to be respectively $\alpha = 1/2$ and $\alpha = 1/3$.

Numerical simulations of the Whitham equation Klein-S. 2014

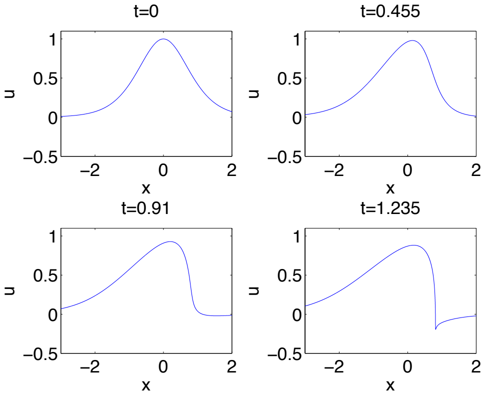
Whitham, $u_0 = -0.1\text{sech}^2 x, t = 20$



Whitham, $u_0 = -\mathrm{sech}^2 x$



Whitham, $u_0 = \operatorname{sech}^2 x$



Conjectures for the Whitham equations

Consider smooth initial data $u_0 \in L_2(\mathbb{R})$ with a single negative hump. Then

- solutions to the Whitham equation and to fKdV equations with $-1 < \alpha < 0$ for initial data u_0 of sufficiently small mass stay smooth for all t and will be radiated away.
- solutions to the Whitham equation (1) and to the fKdV equation with $\alpha = -1/2$ for negative initial data u_0 of sufficiently large mass will develop a cusp at $t^* > t_c$ of the form $|x - x^*|^{1/3}$. The sup norm of the solution remains bounded at the blow-up point.
- solutions to the Whitham equation (1) and to the fKdV equation with $\alpha = -1/2$ for positive initial data u_0 of sufficiently large norm mass will develop a cusp at $t^* < t_c$ of the form $|x - x^*|^{1/2}$.

Open questions and conjectures on the toy model suggested by the numerics

- One expects global well-posedness (with decomposition into solitary waves) in the L^2 subcritical case $\frac{1}{2} < \alpha < 1$ (fKdV).
- One expects blow-up (similar to the L^2 critical or supercritical GKdV regime) when $\frac{1}{3} < \alpha \leq \frac{1}{2}$. Should be hard to prove, in particular for $\frac{1}{3} < \alpha < \frac{1}{2}$ (fKdV).
- One expects blow-up when $0 < \alpha \leq \frac{1}{3}$, of a different nature, but not a shock (fKdV).
- Global well-posedness for small initial data when $0 < \alpha \leq \frac{1}{2}$ (fKdV, Linares-Ponce-S in progress).
- The value $\alpha = \frac{1}{3}$ seems critical in the BBM case.
- Deeper analysis of the original Whitham equation which has (conditionally orbitally stable) solitary waves and blow-up...
- Similar results for other systems for surface or internal water waves, in various regimes. So far we have not found a relevant water waves **system** (say a Boussinesq one) for which the existence time is larger than the *hyperbolic* one $1/\epsilon$.

HAPPY BIRTHDAY WALTER !