Stability of near-resonant gravity-capillary waves

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Acknowledgements

This is joint work with my advisor **Bernard Deconinck** (University of Washington).
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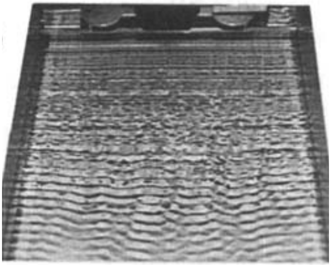
Outline

Why consider surface tension and resonance

Henderson and Hammack (1987) looked at instabilities in the presence of surface tension (resonant triads):

- Consider a tank in deep water
- Generate waves at the back of the tank



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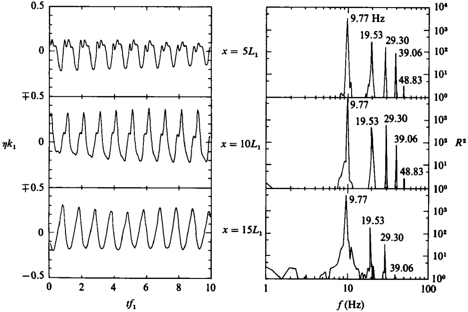

FIGURE 15. Temporal wave profiles and corresponding periodograms for Wilton's ripples (9.8 Hz): $sk_1 = 0.32$, $y = 0$.

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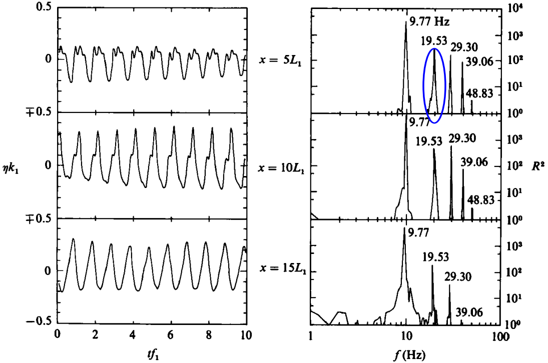
Waves generated at 19.6 Hz excited a harmonic at 9.8 Hz as they propagated



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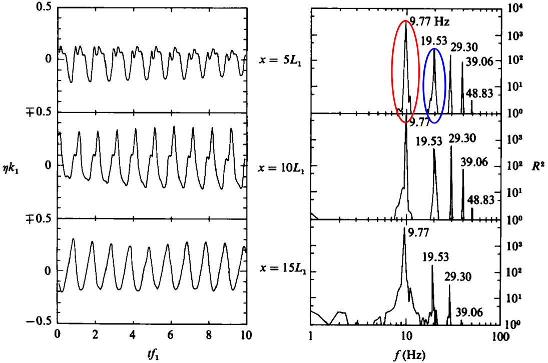

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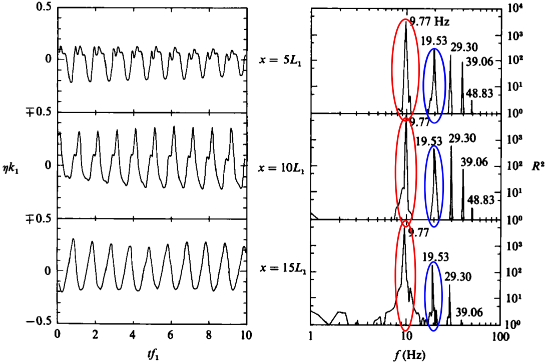

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Some Background

The field of water waves has a long history. A few notable and relevant works in this particular area include

- Wilton (1915) incorporated capillary effects in a series solution and showed it diverges for surface tension parameter equal to $1/n$ (for water of infinite depth).
- Vanden-Broeck et al. (since 1978) - studied the numerical solutions for solitary and periodic capillary-gravity waves with variable surface tension, including Wilton ripples (1D).
- Henderson and Hammack (1987) experimentally observed Wilton ripples in a deep water wave tank.
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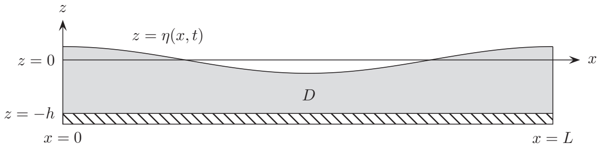
Model

For an inviscid, incompressible fluid with velocity potential $\phi(x, z, t)$



$$\begin{cases} \phi_{xx} + \phi_{zz} = 0, & (x, z) \in D, \\ \phi_z = 0, & z = -h, \\ \eta_t + \eta_x \phi_x = \phi_z, & z = \eta(x, t), \\ \phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = \sigma \frac{\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}, & z = \eta(x, t), \end{cases}$$

where g: gravity, σ: coefficient of surface tension, D: a periodic domain and $\eta(x, t)$: variable surface (in 1D) with period $L = 2\pi$ and depth h.

Approach

Our approach to investigating stability of stationary solutions is a two-step process:

1. Reformulate the problem using the approach by Ablowitz, Fokas and Musslimani and construct solutions for periodic water waves in the travelling frame of reference.
2. Check to see if constructed solutions are spectrally stable by using the Floquet-Fourier-Hill (Bloch) method.

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So far

Gravity waves with and without surface tension are unstable

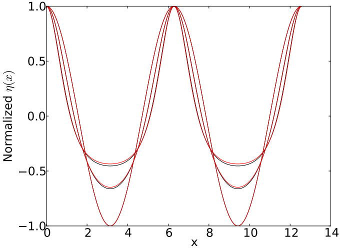

Figure: Eigenvalues of the stability problem for gravity waves with no surface tension (in black) and waves with a small coefficient of surface tension (in red).

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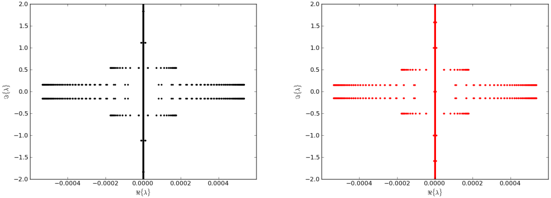

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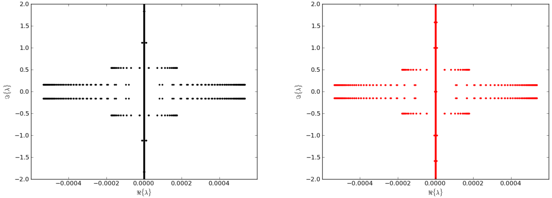

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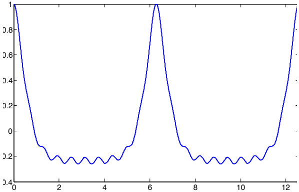
Goal

Examine stability of periodic travelling gravity-capillary water waves near resonance.



Reformulation (Ablowitz, Fokas and Musslimani, 2006)

Starting with Euler's equations

- Setting $q(x,t) = \phi(x, \eta(x,t), t)$ (Zakharov, 1968), the kinematic condition and the Bernoulli equation give

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + \eta_x q_x)^2}{1+\eta_x^2} = \sigma \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}.$$

- Using Laplace's equation and the boundary conditions,

$$\int_0^{2\pi} e^{ikx}\left(i\eta_t \cosh(k(\eta+h)) + q_x \sinh(k(\eta+h))\right) dx = 0,$$

$$\forall k \in \mathbb{Z},\ k \neq 0.$$

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Reformulation

- Switching to the travelling frame by setting $(x,t) \to (x - ct, t)$.
- Looking at the steady-state problem, set $\eta_t = q_t = 0$.
- Use the local equation to obtain q_x.
- The non-local equation becomes

$$\int_0^{2\pi} e^{ikx} \sqrt{(1+\eta_x^2)\left(c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)} \sinh(k(\eta+h))dx = 0$$

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How do we solve this?

1. Stokes' expansion (to see where the resonances are)
2. Numerical continuation employing Newton's method at each step

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Outline

Stokes' Expansion

The algorithm is

1. Set

$$c = \sum_{j=0}^{\infty} \epsilon^j c_j \text{ and } \eta = \sum_{j=0}^{\infty} \epsilon^j \eta_j$$

2. Substitute into

$$\int_0^{2\pi} e^{ikx} \sqrt{(1+\eta_x^2)\left(c^2 - 2g\eta + \frac{2\sigma\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)} \sinh(k(\eta+h))dx = 0$$

3. Group terms by order of ϵ^n
4. Solve the recursion relation such that

$$c_n = f_1(c_{n-1}, c_{n-2}, \ldots, c_0) \text{ and } \eta_n = f_2(\eta_{n-1}, \eta_{n-2}, \ldots, \eta_0)$$

very messy, but explicit!

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A Few Coefficients in Deep Water

In infinite depth ($h = \infty$), obtain

$$c_0 = \sqrt{1+\sigma} \qquad \eta_0 = 0$$

$$c_1 = 0 \qquad \eta_1 = 2\cos(x)$$

$$c_2 = -\frac{2\sigma^2+\sigma+8}{4(1+\sigma)^{1/2}(2\sigma-1)} \qquad \eta_2 = -\frac{2(1+\sigma)}{2\sigma-1}\cos(2x)$$

$$c_3 = 0 \qquad \eta_3 = \frac{3}{2}\frac{2\sigma^2+7\sigma+2}{(3\sigma-1)(2\sigma-1)}\cos(3x)$$

$$\vdots \qquad \vdots$$

Note: blow up if $\sigma = \frac{1}{n}$

Resonance Condition

Isolating for the coefficient of surface elevation in finite depth, we get the following:

$$\left(c_0^2 - (\sigma k^2 + g)\tanh(kh)\right)\hat{\eta}_k = \text{"a mess"}$$

Resonance if

$$\sigma = \frac{g}{k}\left(\frac{\tanh(hk) - k\tanh(h)}{\tanh(h) - k\tanh(hk)}\right) \text{ with } k \in \mathbb{Z}$$

Near resonance (small divisor problem) if

$$c_0^2 - (\sigma k^2 + g)\tanh(kh) \approx 0 \text{ with } c_0 = \sqrt{(\sigma + g)\tanh(h)}$$

Fix g and h, solve for σ with a variety of k values near 20 or near 10.

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Numerical Continuation

Recall

$$\int_0^{2\pi} e^{ikx} \sqrt{(1+\eta_x^2)\left(c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)} \sinh(k(\eta+h))dx = 0.$$

We want to generate a bifurcation diagram:

1. Assume in general $\eta_N(x) = \sum_{j=1}^{N} a_j \cos(jx)$.
2. Linearizing we can find the bifurcation will start when $c = \sqrt{(g+\sigma)\tanh(h)}$ and $\eta(x) = a\cos(x)$.
3. Use this guess in Newton's method to compute the true solution.
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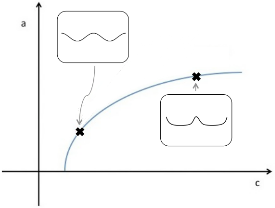


Near Resonant Solutions - near $k = 20$

Let $h = 0.05$ and compute σ for $k = 20.5$

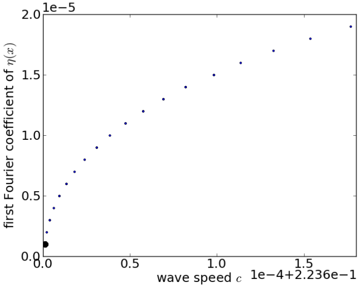


Figure: Bifurcation branch

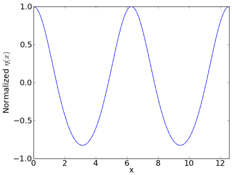


Figure: Physical profile of the wave

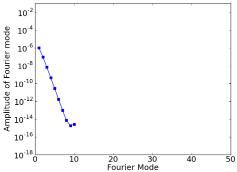


Figure: Fourier coefficients of the profile

Let $h = 0.05$ and compute σ for $k = 20.05$

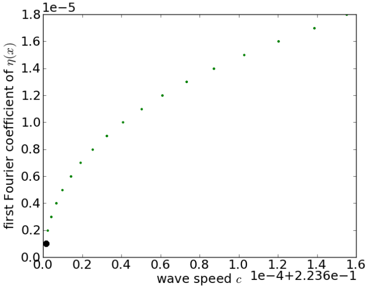
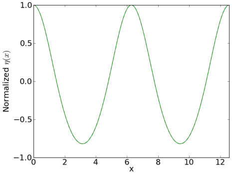


Figure: Bifurcation branch



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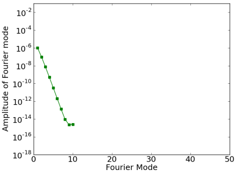


Figure: Fourier coefficients of the profile

Let $h = 0.05$ and compute σ for $k = 10.5$

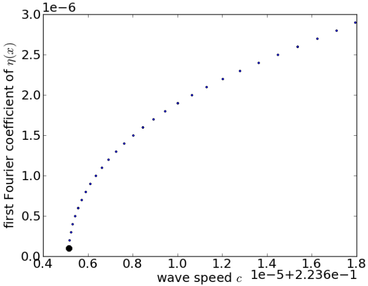
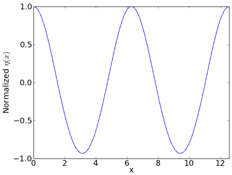


Figure: Bifurcation branch



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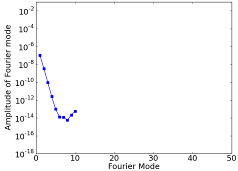


Figure: Fourier coefficients of the profile

Near Resonant Solutions - near $k = 10$

Let $h = 0.05$ and compute σ for $k = 10.05$

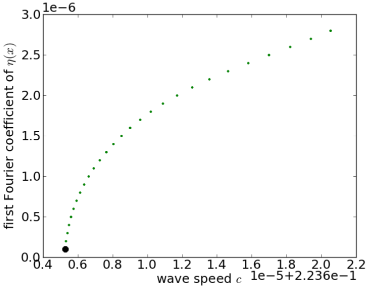
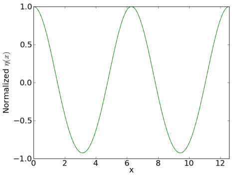


Figure: Bifurcation branch



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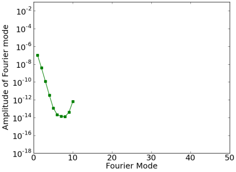

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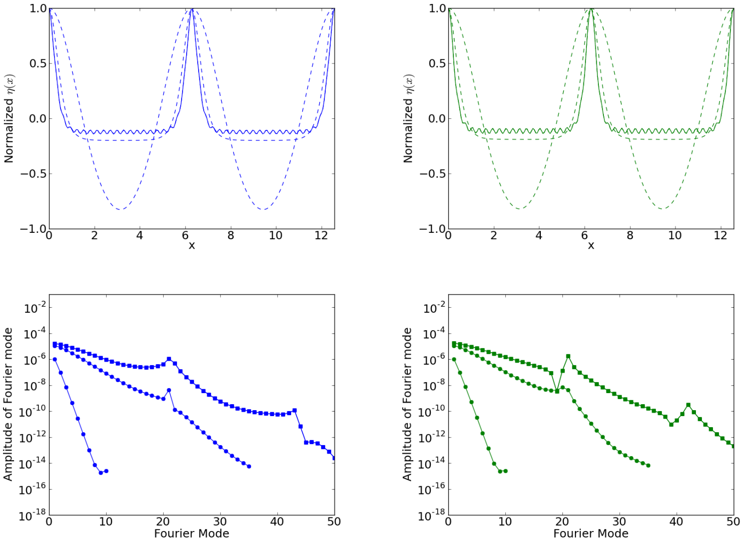
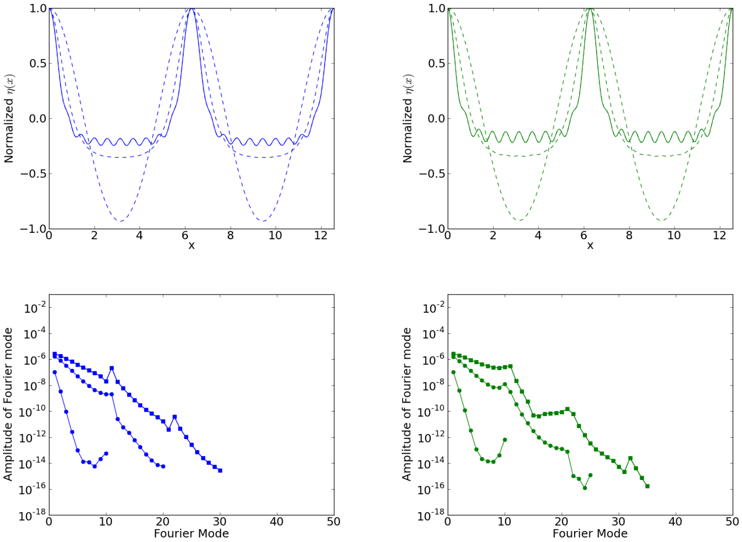
Comparisons of Profiles - near $k = 20$

$\sigma \approx 7.80 \times 10^{-4}$ ($k = 20.5$) $\sigma \approx 7.82 \times 10^{-4}$ ($k = 20.05$)



Comparisons of Profiles - near $k = 10$

$\sigma \approx 8.18 \times 10^{-4}$ $(k = 10.5)$ $\sigma \approx 8.19 \times 10^{-4}$ $(k = 10.05)$



Outline

Stability Eigenvalue Problem

Recall the local equation

$$q_t - cq_x + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t - c\eta_x + q_x\eta_x)^2}{1+\eta_x^2} = \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}$$

and the nonlocal equation

$$\int_0^{2\pi} e^{ikx}\left[i(\eta_t - c\eta_x)\cosh(k(\eta+h)) + q_x\sinh(k(\eta+h))\right]dx = 0.$$

1. Let $q(x,t) = q_0(x) + \epsilon q_1(x)e^{\lambda t} + \ldots$ and $\eta(x) = \eta_0(x) + \epsilon\eta_1(x)e^{\lambda t} + \ldots$.
2. Using Floquet decompositions, we set $\eta_1 = e^{i\mu x}\tilde{\eta}_1$ and $q_1 = e^{i\mu x}\tilde{q}_1$.
3. Apply Fourier decomposition with $\tilde{\eta}_1 = \sum_{m=-\infty}^{\infty}\hat{N}_m e^{imx}$ and $\tilde{q}_1 = \sum_{m=-\infty}^{\infty}\hat{Q}_m e^{imx}$.
4. To allow perturbations of a different period, introduce spatial averaging in the nonlocal equation.

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Eigenvalue Problem

After all the substitutions, obtain

$$\Rightarrow \begin{bmatrix} S & T \\ U & V \end{bmatrix} \begin{pmatrix} \hat{N} \\ \hat{Q} \end{pmatrix} = \lambda \begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \begin{pmatrix} \hat{N} \\ \hat{Q} \end{pmatrix}$$

The local equation gives the row in blue and the nonlocal equation gives the row in green.

Generalized eigenvalue problem

$$\lambda = \lambda(\mu, m, \sigma)$$

The problem is Hamiltonian and due to symmetries,

$\mathbb{R}\{\lambda\} \neq 0 \Rightarrow$ instability.

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Instability

For flat water, can compute the eigenvalues explicitly

$$\lambda^{\pm}_{\mu+m} = ic(\mu+m) \pm i\sqrt{[g(\mu+m)+\sigma(\mu+m)^3]\tanh((\mu+m)h)}$$

$\Rightarrow$ flat water is spectrally stable

How does an instability arise?

- Eigenvalues are continuous with respect to the wave amplitude
- As amplitude increases they may develop a non-zero real part

A necessary condition for loss of stability is

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Instabilities near $k = 20$

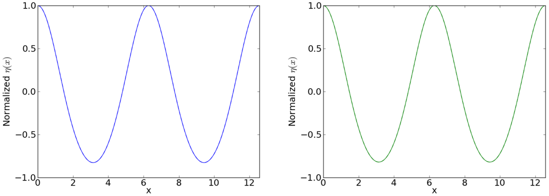

Figure: Wave profile

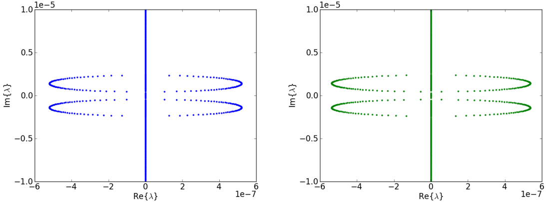

Figure: Eigenvalues in the complex plane

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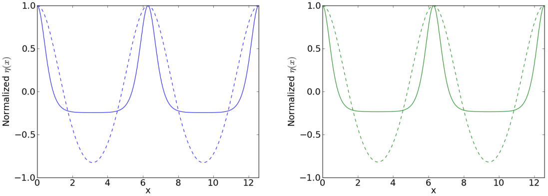

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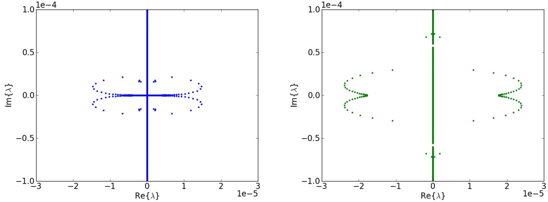

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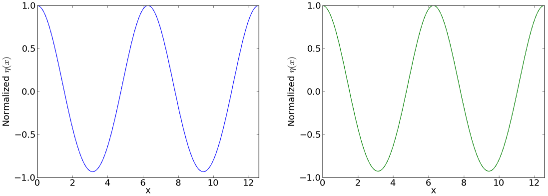
Instabilities near $k = 10$



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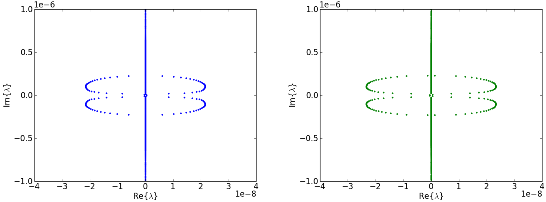

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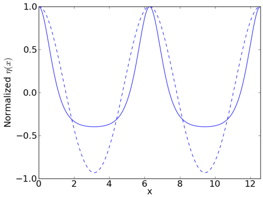

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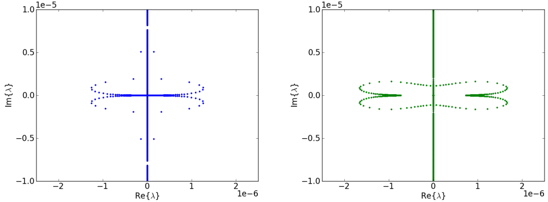

Figure: Eigenvalues in the complex plane

Conclusions

- Solutions can be computed near resonance.
- A larger coefficient of surface tension does not stabilize the solutions.
- As the parameter of surface tension gets larger, the waves become more unstable.

Future Work

- Compute solutions to a higher precision (quadruple precision, with Jon Wilkening at Berkeley).
- Compute the stability spectra for more values of the Floquet parameter.
- Track the new instabilities along the bifurcation branch.
- Track the instabilities as the surface tension parameter is varied.
- Examine the form of the perturbations that lead to the new instabilities.

Thank you for your attention