Invariant Variational Calculus

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Ingredients

- Invariant Euler-Lagrange operator

  Invariant Euler-Lagrange Equations and the Invariant Variational Bicomplex, I. Kogan, P. Olver, Acta Appl. Math. 76, 137-193, (2003)

- Invariant Noether correspondence (work in progress)

- Symbolic implementation (iVB package) *using* MAPLE *package* VESSIOT *for calculus on the jet bundles.* by I. Anderson et al.

  Needs translation to DIFFERENTIALGEOMETRY package !!!

Euler's Elastica

What is the shape of a thin elastic rod of a fixed length with fixed end-points and tangent directions at the end-points?

Find $\gamma(t) = (x(t), y(t))$ that minimizes bending energy:

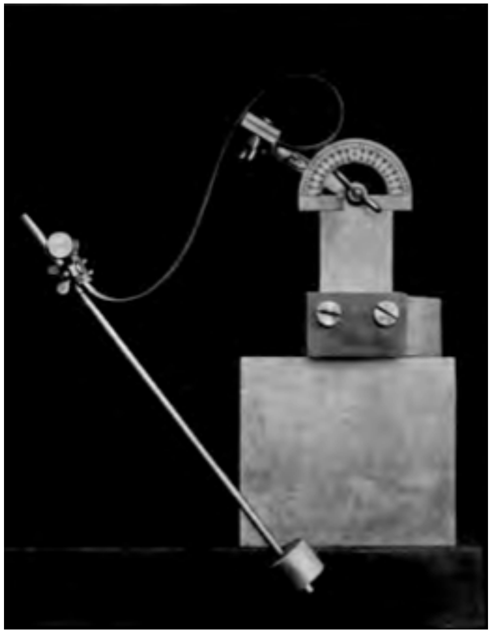
$$\boxed{\mathcal{L}(\gamma) = \frac{1}{2}\int_0^l \kappa^2\, ds,}$$

$\kappa = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}$ is Euclidean curvature and $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$ is the infinitesimal arclength.

This variational problem is invariant under the group of rigid motions on the plane ($E(2) = O(2) \ltimes \mathbb{R}^2$).

Max Born's Ph.D thesis, 1906,

"Investigations of the stability of the elastic line in the plane and in space under different boundary conditions":



- Max Born. *Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum, under verschiedenen Grenzbedingungen*. PhD thesis, University of Göttingen, 1906.
- R. Levien. The elastica: a mathematical history, 2008. `http://www.eecs.berkeley.edu/Pubs/TechRpts/2008/EECS-2008-103.pdf`

Euler-Lagrange equations for Euler's Elastica

Let γ be parametrized by x-variable: $\gamma = (x, u(x))$, then

$$\frac{1}{2}\int_0^l \kappa^2\, ds = \frac{1}{2}\int_a^b \frac{u_{xx}^2}{(1+u_x^2)^{5/2}} dx.$$

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Notation: $u_1 = u_x, \ldots, u_4 = u_{xxxx}$.

$$L = \tfrac{1}{2}\tfrac{u_2^2}{(1+u_1^2)^{5/2}}$$

$$\downarrow E = \sum_k (-1)^k \left(\frac{d}{dx}\right)^k \frac{\partial}{\partial u_k} = \frac{\partial}{\partial u} - \left(\frac{d}{dx}\right)\frac{\partial}{\partial u_1} + \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_2}$$

$$\frac{2\, u_4\, (u_1^2+1)^2 + 5\, u_2^3\, (6\, u_1^2-1) - 20\, u_1\, u_2\, u_3\, (u_1^2+1)}{(u_1^2+1)^{9/2}} = 0$$

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$u_1 = u_x, \ldots, u_4 = u_{xxxx}$ $\qquad$ $\kappa_s = \frac{d\kappa}{ds}, \kappa_{ss} = \frac{d\kappa_s}{ds}, \ldots$

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$$\Big\downarrow \; \frac{\partial}{\partial u} - \left(\frac{d}{dx}\right)\frac{\partial}{\partial u_1} + \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_2}$$

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$$L = \frac{1}{2}\frac{u_2^2}{(1+u_1^2)^{5/2}} \quad \Longleftrightarrow \quad \tilde{L} = \frac{1}{2}\kappa^2$$

$$\downarrow E = \frac{\partial}{\partial u} - \left(\frac{d}{dx}\right)\frac{\partial}{\partial u_1} + \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_2} \qquad \downarrow$$

?(not $\frac{\partial}{\partial \kappa}$!!!)

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G-Invariant Euler-Lagrange operators for planar curves

- A Lie group G acts on (x, u)-space $\to$ action on planar curves.

- κ is a (lowest order) differential invariant (G-curvature);

- ds is a (lowest order) G-invariant one-form (G-arc-length form);

- G-invariant total derivative $\mathcal{D} = \frac{d}{ds}$; $\kappa_i = \mathcal{D}^i \kappa$.

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G-symmetric variational problem: $\int \tilde{L}\,(\kappa, \kappa_1, \ldots, \kappa_n)\; ds.$

- Express Euler-Lagrange operator in terms of κ and $\mathcal{D}$.

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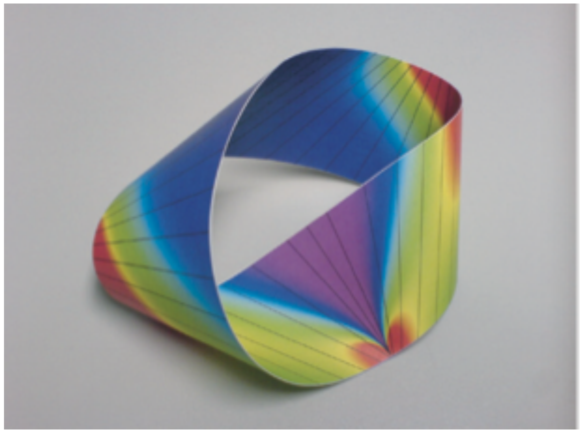
G-symmetric variational problem: $\int \tilde{L}\,(\kappa, \kappa_1, \ldots, \kappa_n)\; ds.$

- Express Euler-Lagrange operator in terms of κ and $\mathcal{D}$.
- Generalize to G-symmetric variational problem in higher dimensions (several dependent and independent variables)

"The shape of a Möbius strip", Starostin and Van der Heijden, Nature Materials. 2007.



The shape of a Möbius strip is determined by its centerline $\gamma(s)$, which minimizes:

$$\mathcal{L}(\gamma) = \frac{1}{w} \int_0^l \frac{(\kappa^2 + \tau^2)^2}{\kappa \tau_s - \tau \kappa_s} \ln \left(\frac{\kappa^2 + w(\kappa \tau_s - \tau \kappa_s)}{\kappa^2 - w(\kappa \tau_s - \tau \kappa_s)} \right) ds$$

$2w$ is the width of the strip, κ is the curvature and τ is the torsion of γ.

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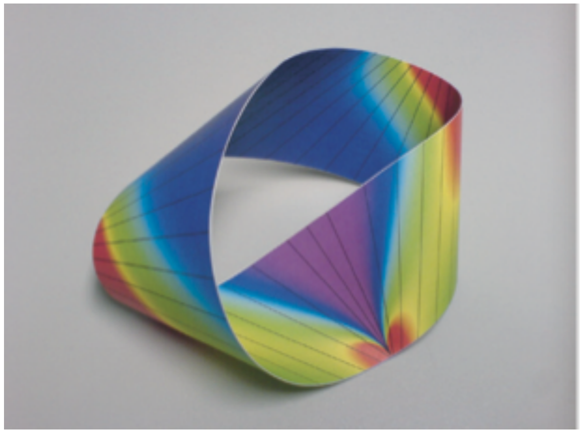

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This variational problem is invariant under the group of rigid motions in $\mathbb{R}^3$ ($E(3) = O(3) \ltimes \mathbb{R}^3$).

Minimal surfaces

Find $u(x,y)$, s. t. the surface $z = u(x,y)$ with a fixed boundary has the minimal area:

$$\boxed{\mathcal{L}(u) = \int_D \sqrt{u_x^2 + u_y^2 + 1}\; dx \wedge dy = \int_S \omega,}$$

$\omega = \sqrt{u_x^2 + u_y^2 + 1}\; dx \wedge dy$ infinitesimal area (Euclidean invariant).

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$$L = \sqrt{u_x^2 + u_y^2 + 1} \qquad \Longleftrightarrow \qquad \tilde{L} = 1$$

$$\Big\downarrow\; E = \frac{\partial}{\partial u} - \left(\frac{d}{dx}\right)\frac{\partial}{\partial u_x} - \left(\frac{d}{dy}\right)\frac{\partial}{\partial u_y} \qquad\qquad \Big\downarrow \quad ?$$

$$0 = \frac{1}{2}\frac{u_{xx}\,(u_y^2+1)^2 + u_{yy}\,(u_x^2+1)^2 - 2\,u_x\,u_y\,u_{xy}}{(u_x^2+u_y^2+1)^{3/2}} \qquad = \qquad \text{mean curvature}$$

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Results

- Euler-Lagrange operators for variational problems for plane and space curves and surfaces symmetric under Euclidean transformations appeared in
  - Griffiths (1983), Anderson (1989)

- General formula for any number of dependent and independent variables first appeared in
  - IK and Olver, (2001, 2003) and somewhat less explicitly in Itskov (2002).

General formula for planar curves $\int \tilde{L}\left(\kappa, \kappa_1, \ldots, \kappa_n\right) ds$.

$$\boxed{\tilde{E} = \mathcal{A}^* \circ \mathcal{E} - B^* \circ \mathcal{H}}$$

$$\mathcal{E}\left(\tilde{L}\right) = \sum_{i=0}^{n} (-\mathcal{D})^i \frac{\partial \tilde{L}}{\partial \kappa_i}, \qquad \mathcal{H}\left(\tilde{L}\right) = \sum_{i>j\geq 0}^{n} \kappa_{i-j} (-\mathcal{D})^j \frac{\partial \tilde{L}}{\partial \kappa_i} - \tilde{L}.$$

- invariant differential operators $\mathcal{A}$ and $\mathcal{B}$ are measuring infinitesimal variation of κ and ds in an invariant "normal" direction, respectively.

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- invariant differential operators $\mathcal{A}$ and $\mathcal{B}$ are measuring infinitesimal variation of κ and ds in an invariant “normal” direction, respectively.

- $\mathcal{A}$ and $\mathcal{B}$ are algorithmically computable from the structure equations of an invariant coframe and infinitesimal generators of the group action.

General formula for planar curves $\int \tilde{L}(\kappa, \kappa_1, \ldots, \kappa_n)\, ds$.

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- invariant differential operators $\mathcal{A}$ and $\mathcal{B}$ are measuring infinitesimal variation of κ and ds in an invariant "normal" direction, respectively.

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- if we have p dependent and q independent variables then we have a similar formula, with scalar differential operators replaced with vector and matrix operators of appropriate dimensions.

Variational problems for planar curves $(x, u(x))$

- **Euclidean group:** $SE(2) = SO(2) \ltimes R^2$.

$$\boxed{\kappa = \frac{u_2}{(1+u_1^2)^{3/2}}, \qquad ds = \sqrt{1 + u_1^2}\, dx}$$

$$\mathcal{A} = \mathcal{A}^* = \left(\frac{d}{ds}\right)^2 + \kappa^2$$

$$\mathcal{B} = \mathcal{B}^* = -\kappa$$

- **Affine group:** $SA(2) = SL(2) \ltimes R^2$

$$\boxed{\mu = \frac{u_2\, u_4 - \frac{5}{3} u_3^2}{u_2^{8/2}}, \qquad da = u_2^{1/3}\, dx}$$

$$\mathcal{A} = \mathcal{A}^* = \left(\frac{d}{da}\right)^4 + \frac{5}{3}\mu \left(\frac{d}{da}\right)^2 + \frac{5}{3}\mu_a \left(\frac{d}{da}\right) + \frac{1}{3}\mu_{aa} + \frac{4}{9}\mu^2$$

$$\mathcal{B} = \mathcal{B}^* = \frac{1}{3}\left(\frac{d}{da}\right)^2 - \frac{2}{9}\mu$$

Variational calculus can be done in the context of variational bicomplex

Dedecker (1957), Tulczyjew (1977), Tsujishita (1982), Takens (1979), Vinogradov (1984), Anderson (1989), ...

Invariant variational calculus can be done in the context of invariant variational bicomplex

Anderson (1989), Anderson and Pohjanpelto (1995), Kogan and Olver (2001,2003), Itskov (2002), Thompson and Valiquette (2011)

Equivariant moving frame method by Fels and Olver (1999) gives rise to invariant variational bicomplex with computable structure.

Standard local coframe on $\mathbb{J}^\infty(M,p)$

Local coordinates: $x^1,\ldots,x^p,u^1,\ldots,u^q,u^m_J$, $m=1\ldots q$, J – multi-index.

Basis of horizontal sub-bundle	Basis of vertical sub-bundle
Cotangent	
horizontal one-forms	contact one-forms ($m=1,\ldots,q$)
$dx^1,\ldots,dx^p$	$\theta^m = du^m - \sum_{i=1}^{p} u^m_i\,dx^i,$ $\theta^m_J = du^m_J - \sum_{i=1}^{p} u^m_{Ji}\,dx^i.$
Tangent	
total derivatives:	vertical derivatives
$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{m=1}^{q} u^m_i \frac{\partial}{\partial u^m}$ $+\sum_{m,J} u^m_{Ji}\frac{\partial}{\partial u^m_J}$	$\frac{\partial}{\partial u^m},\quad \frac{\partial}{\partial u^m_J}.$

Bigrading of exterior differential algebra

Grading: $\Lambda^* = \bigoplus \Lambda^k$, where $\Lambda^k = \left\{ \sum \underbrace{\text{one form} \wedge \cdots \wedge \text{one form}}_{k \text{ times}} \right\}$.

$$d\colon \Lambda^k \to \Lambda^{k+1}, \quad d \circ d = 0 \Longrightarrow \text{ de Rham complex.}$$

Bigrading: $\Lambda^* = \bigoplus \Lambda^{s,t}$, where

$$\Lambda^{s,t} = \left\{ \sum \underbrace{\text{hor. 1-form} \wedge \cdots \wedge \text{hor. 1-form}}_{s \text{ times}} \wedge \underbrace{\text{cont. 1-form} \wedge \cdots \wedge \text{cont. 1-form}}_{t \text{ times}} \right\}$$

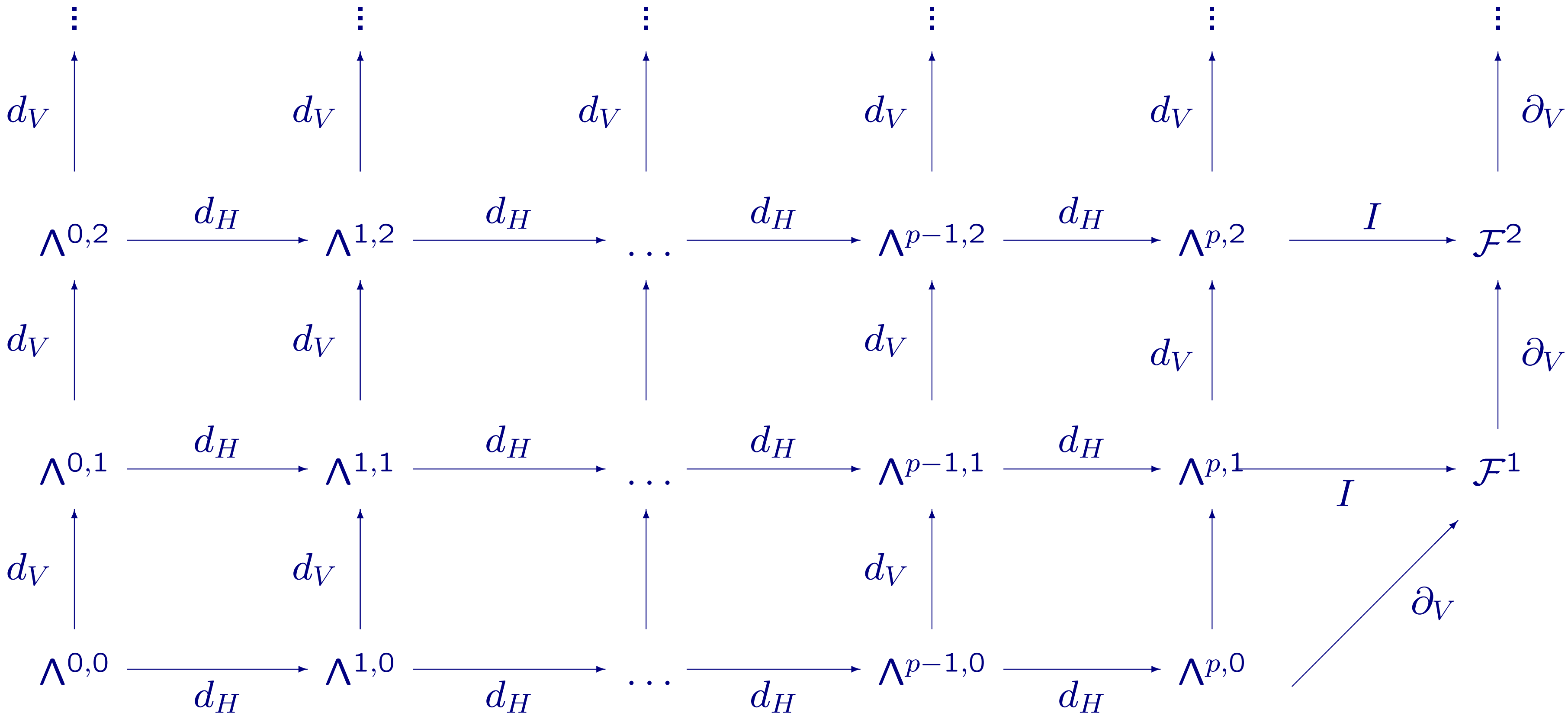
$$d\colon \Lambda^{s,t} \to \Lambda^{s+1,t} \oplus \Lambda^{s,t+1} \quad \Rightarrow \quad d = d_H + d_V$$

$$d^2 = (d_H + d_V)^2 = 0 \quad \Rightarrow \quad \boxed{d_H^2 = 0, \quad d_V^2 = 0, \quad d_H \circ d_V = -d_V \circ d_H}$$

$$\Downarrow$$

Bicomplex

Variational Bicomplex (locally exact)



$I: \Lambda^{p,s} \longrightarrow \mathcal{F}^s = \Lambda^{p,s}/Im\, d_H$ - integration by parts operator

$\partial_V = I \circ d_V$- variational derivative;

$$d_V d_H = -d_H d_V, \quad d_H^2 = 0, \quad d_V^2 = 0, \quad \partial_V^2 = 0, \quad I \circ d_H = 0, \quad \partial_V \circ d_H = 0.$$

Integration by parts operator

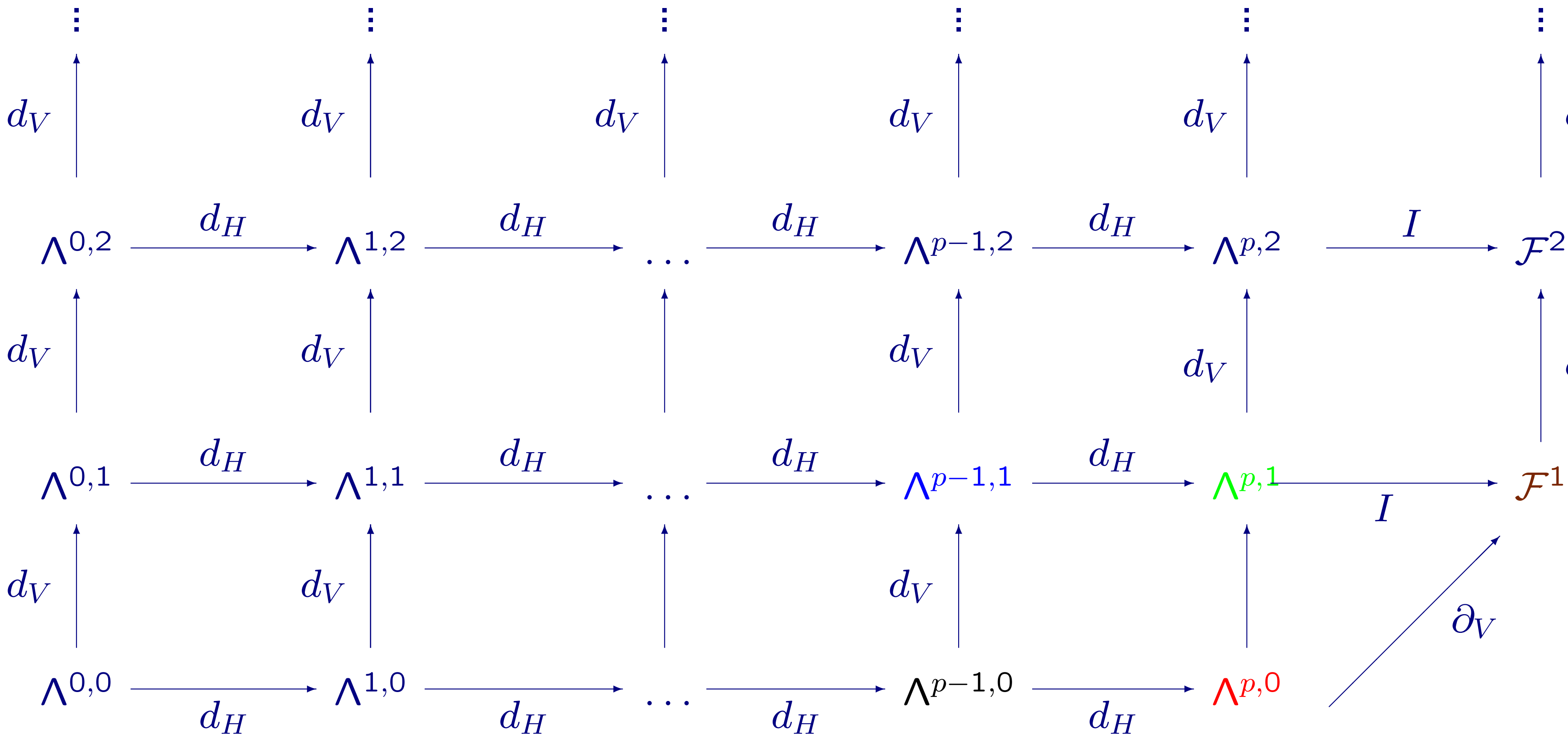
For $\Omega \in \Lambda^{p,s}$, $s > 0$.

$$I(\Omega) = \frac{1}{s} \sum_{m=1}^{q} \theta^m \wedge \left(\sum_J (-1)^{|J|} \frac{d}{dx_J} \left(\frac{\partial}{\partial u_J^m} \lrcorner \Omega \right) \right).$$

$$\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} \xrightarrow{\quad d_V \quad} \sum_{m,J} \frac{\partial L}{\partial u_J^m} \theta_J^m \wedge d\mathbf{x} \xrightarrow{\quad I \quad} \sum_{m,J} E_m(L)\, \theta^m \wedge d\mathbf{x}$$

$E_m, \quad m = 1, \ldots, q$ are Euler-Lagrange operators.

$E_m(L) = 0, \quad m = 1, \ldots, q$ are Euler-Lagrange equations.



- $\lambda = L d\mathbf{x} \in \Lambda^{p,0}$ - Lagrangian; $\partial_V \lambda = \sum_{m=1}^{q} E_m(L)\, \theta^m \wedge d\mathbf{x}$, ($E_m(L) = 0$ are E.-L. eq.).
- $d_V \lambda - \partial_V \lambda = d_H \nu, \quad \nu \in \Lambda^{p-1,1}$

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & \uparrow\partial_V \\
\Lambda^{0,2} & \xrightarrow{d_H} & \Lambda^{1,2} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,2} & \xrightarrow{d_H} & \Lambda^{p,2} & \xrightarrow{I} & \mathcal{F}^2 \\
d_V\uparrow & & d_V\uparrow & & \uparrow & & d_V\uparrow & & d_V\uparrow & & \uparrow\partial_V \\
\Lambda^{0,1} & \xrightarrow{d_H} & \Lambda^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & {\color{blue}\Lambda^{p-1,1}} & \xrightarrow{d_H} & {\color{green}\Lambda^{p,1}} & \xrightarrow[I]{} & \mathcal{F}^1 \\
d_V\uparrow & & d_V\uparrow & & \uparrow & & d_V\uparrow & & \uparrow & \nearrow\partial_V & \\
\Lambda^{0,0} & \xrightarrow[d_H]{} & \Lambda^{1,0} & \xrightarrow[d_H]{} & \dots & \xrightarrow[d_H]{} & \Lambda^{p-1,0} & \xrightarrow[d_H]{} & {\color{red}\Lambda^{p,0}} & &
\end{array}$$

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- $d_V\lambda - \partial_V\lambda = d_H\nu, \quad \nu \in \Lambda^{p-1,1}$
- v.-f. $\mathbf{v}$ is an infinitesimal variational symmetry if $\exists \alpha \in \Lambda^{p-1,0}$ s.t. $\boxed{\mathbf{v}^\infty(\lambda) = d_H(\alpha)}$.
- Noether correspondence: $\boxed{\pi = \mathbf{v}^\infty \lrcorner \nu + \alpha}$ is a conservation law: $\boxed{d_H\pi = 0 \text{ mod } \{E_m(L)\}}$.

Invariantization (Fels and Olver (1999))

Theorem: Let $\mathfrak{g}$ be an r-dim'l Lie algebra of infinitesimal transformations on $\mathbb{J}^0$. Then for some $k_0 \leq r$, $\exists$ submanifold $\mathcal{K} \subset \mathbb{J}^n$ of codimension r (called local cross-section) such that

$$T|_z\mathcal{K} \bigoplus \mathfrak{g}|_z = T|_z\mathbb{J}^{k_0},\ \forall z \in \mathcal{K}.$$

$\mathcal{K}$ can be lifted to $\mathbb{J}^k$ for all $k \geq k_0$.

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Theorem: $\mathcal{K}$ defines invariantization ι for functions, differential forms and vector fields on an open neighborhood of $\mathcal{K}$:

- $\forall f \in C^\infty(\mathbb{J}^\infty)$ $\exists!\mathfrak{g}$-invariant ιf s.t. $\iota f|_{\mathcal{K}} = f|_{\mathcal{K}}$.
- $\forall \Omega \in \Lambda^*(\mathbb{J}^\infty)$ $\exists!\mathfrak{g}$-invariant $\iota\Omega$ s.t. $\iota\Omega|_{\mathcal{K}} = \Omega|_{\mathcal{K}}$.
- $\forall \mathbf{w} \in \mathcal{T}(\mathbb{J}^\infty)$ $\exists!\mathfrak{g}$-invariant $\iota\mathbf{w}$ s.t. $\iota\mathbf{w}|_{\mathcal{K}} = \mathbf{w}|_{\mathcal{K}}$.

Properties of ι

- ι preserves linear independence of differential forms and vector-fields
- ι preserves contact-ideal
- structure equations for invariantized frame and coframe are algorithmically computable without explicit formulas for invariants!!:

$$\boxed{d\,(\iota\Omega) = \iota\,(d\,\Omega) - \sum_{\kappa=1}^{r} \iota\left[dK \cdot \mathbf{v}(K)^{-1}\right] \wedge \iota\left[\mathbf{v}_{\kappa}\,(\Omega)\right]}$$

$\mathbf{v} = (\mathbf{v}_1, \ldots \mathbf{v}_r)$ – is a basis of $\mathfrak{g}$.

$K = (K^1, \ldots, K^r)$ – is a row vector of functions, whose zero-set defines the cross-section $\mathcal{K}$.

$\mathbf{v}(K)$ is an $r \times r$-matrix whose (i, j)-th entry equals to $\mathbf{v}_j^{\infty}(K^i)$.

Invariant local frame and coframe on $\mathbb{J}^\infty$

$i = 1, \ldots, p, \quad m = 1 \ldots q, \quad J$ – symmetric multi-index.

Invariant horizontal basis	Invariant vertical basis
Tangent	
invariant total diff. operators $\mathcal{D}_i = \iota\left(\frac{d}{dx^i}\right)$ span $\{\mathcal{D}_i\}$ = span $\left\{\frac{d}{dx_i}\right\}$	invariant vertical diff. operators $\mathcal{V}^J_m = \iota\left(\frac{\partial}{\partial u^m_J}\right)$ span $\{\mathcal{V}_m\} \neq$ span $\left\{\frac{\partial}{\partial u^m}\right\}$ unless the action is projectable
Cotangent	
invariant "horizontal" one-forms $\omega^i = \iota(dx^i)$ span $\left\{\omega^i\right\} \neq$ span $\left\{d\,x^i\right\}$ unless the action is projectable	invariant contact one-forms $\vartheta^m_J = \iota(\theta^m_J)$ span $\left\{\vartheta^m_J\right\}$ = span $\left\{\theta^m_J\right\}$

Example $\mathfrak{se}(2)$-invariant frame and coframe $\mathbb{J}^\infty(\mathbb{R}^2, 1)$

Invariant horizontal basis	Invariant vertical basis
Tangent	
invariant total diff. operators $\mathcal{D} = \frac{1}{\sqrt{1+u_x^2}} \frac{d}{dx} = \frac{d}{ds}$	invariant vertical diff. operators $\mathcal{V} = -\frac{u_x}{\sqrt{1+u_x^2}} \frac{\partial}{\partial x} + \frac{1}{\sqrt{1+u_x^2}} \frac{\partial}{\partial u}$ $\mathcal{V}^x = (1 + u_x^2)\, \frac{\partial}{\partial u_x} + 3\, u_x u_{xx}\, \frac{\partial}{\partial u_{xx}}$ $\dots$
Cotangent	
invariant “horizontal” one-forms $\omega = ds + \frac{u_x}{\sqrt{1+u_x^2}} \theta,$ where $ds = \sqrt{1 + u_x^2} dx$	invariant contact one-forms $\vartheta = \frac{\theta}{\sqrt{1+u_x^2}}$ $\vartheta_x = \frac{(1+u_x^2)\, \theta_x - u_x u_{xx} \theta}{(1+u_x^2)^2}$ $\dots$

Invariant bigrading: $\Lambda^* = \oplus\, \tilde{\Lambda}^{s,t}$

Unless the action is projectable $\tilde{\Lambda}^{s,t} \neq \Lambda^{s,t}$ and for $s \geq 1$:

$$d\colon \tilde{\Lambda}^{s,t} \longrightarrow \tilde{\Lambda}^{s+1,t} \oplus \tilde{\Lambda}^{s,t+1} \oplus \tilde{\Lambda}^{s-1,t+2} \Rightarrow d = d_{\tilde{H}} + d_{\tilde{V}} + d_W$$

$$d^2 = (d_{\tilde{H}} + d_{\tilde{V}} + d_W)^2 = 0$$

$$\boxed{d_{\tilde{H}}^2 = 0, \quad d_{\tilde{V}}^2 + d_{\tilde{H}} d_W + d_W d_{\tilde{H}} = 0, \quad d_{\tilde{H}} \circ d_{\tilde{V}} = -d_{\tilde{V}} \circ d_{\tilde{H}}, \quad d_W^2 = 0}$$

$$\boxed{\partial_{\tilde{V}}^2 \neq 0!!}$$

Generating set of invariants

- Tresse (1894): Differential algebra of invariants is finitely generated.

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$m = 1, \ldots, q, \quad J = (j_1, \ldots, j_l)$ is a $\boxed{\text{symmetric}}$ multi-index contains a finite set of invariants

$$\{\kappa^1, \ldots, \kappa^{\tilde{q}}\}$$

such that any other invariant function on $\mathbb{J}^\infty$ can be expressed as a function of $(\kappa^l_{\hat{J}})$, where $l = 1, \ldots, \tilde{q}$, $\hat{J} = (j_1, \ldots, j_l)$ is a $\boxed{\text{non-symmetric}}$ multi-index and $\boxed{\kappa^l_{\hat{J}} = \mathcal{D}_{j_l} \cdots \mathcal{D}_{j_1} \kappa^l}$.

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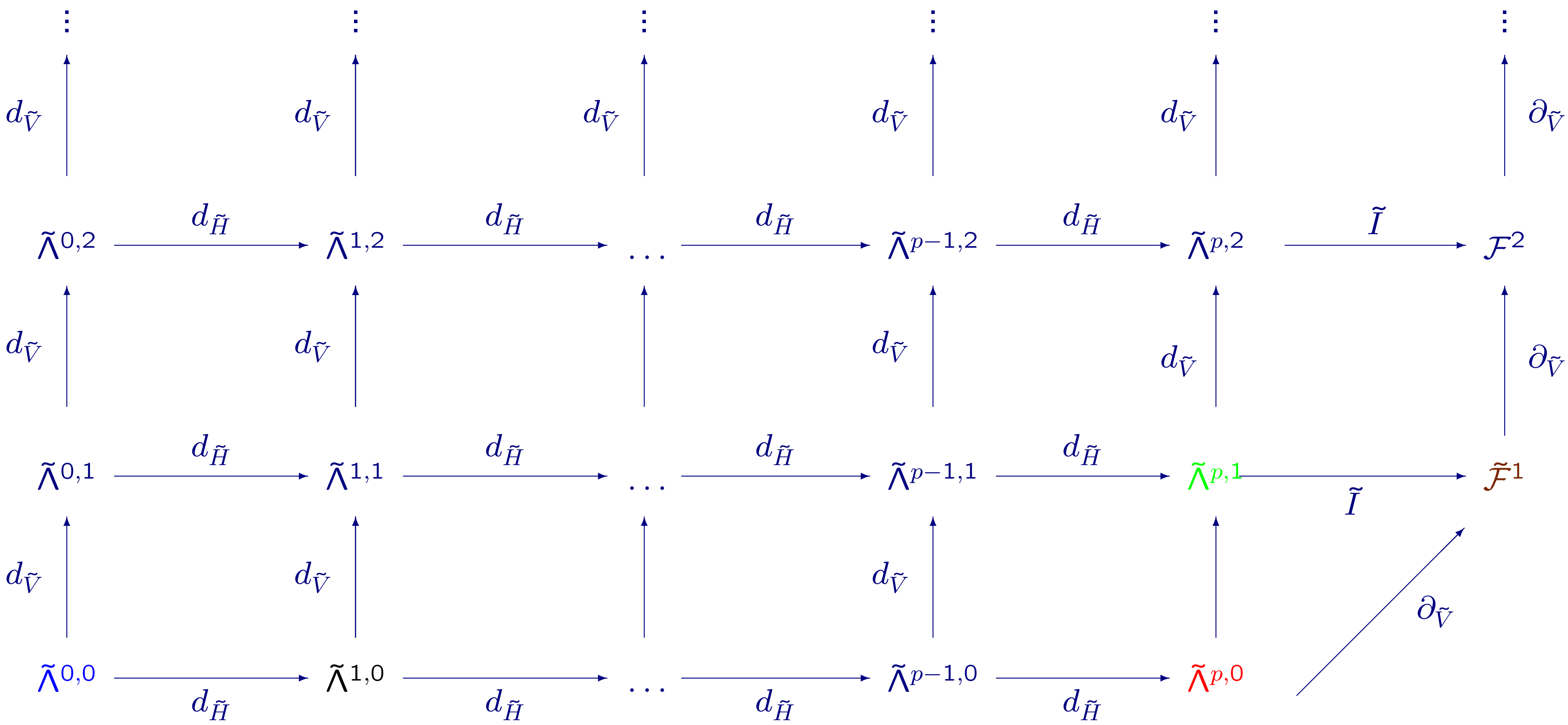
such that any other invariant function on $\mathbb{J}^\infty$ can be expressed as a function of $(\kappa^l_{\hat{J}})$, where $l = 1, \ldots, \tilde{q}, \ \hat{J} = (j_1, \ldots, j_l)$ is a non-symmetric multi-index and $\boxed{\kappa^l_{\hat{J}} = \mathcal{D}_{j_l} \cdots \mathcal{D}_{j_1} \kappa^l}$.

- A symmetric variational problem can be represented by $\boxed{\tilde{\lambda} = \tilde{L}[\kappa]\,\omega,}$ where $[\kappa] = \{\kappa^l_{\hat{J}} | l \in \{1, \ldots, \tilde{q}\}\}, \quad \omega = \omega^1 \wedge \cdots \wedge \omega^p$.

Invariant Variational "Bicomplex". • variational edge is a complex



- $\tilde\lambda = \tilde L[\kappa]\,\omega$ - Lagrangian $\qquad \partial_{\tilde V}\tilde\lambda = \sum_{m=1}^{q} \tilde E_m(\tilde L)\,\vartheta^m \wedge \omega, \quad$ ($\tilde E_m(\tilde L) = 0$ are E.-L. eq.).
- $\tilde E_m = \sum_{l=1}^{\tilde q} \mathcal{A}^{*l}_{\ m}\mathcal{E}_l - \sum_{i,j=1}^{p} \mathcal{B}^{*j}_{\ im}\mathcal{H}^i_j,$ where

$$d_{\tilde V}(\kappa^l) = \sum_{m=1}^{q} \mathcal{A}^l_m(\vartheta^m) \quad \text{and} \quad d_{\tilde V}(\omega^j) = \sum_{i=1}^{p}\sum_{m=1}^{q} \mathcal{B}^j_{im}(\vartheta^m)\wedge\omega^i$$

- $\tilde{\lambda} = \tilde{L}[\kappa]\,\omega$, where $[\kappa] = \{\kappa^l_{\hat{J}} | l \in \{1,\ldots,\tilde{q}\}\}, \quad \omega = \omega^1 \wedge \cdots \wedge \omega^p.$

- $\partial_V \tilde{\lambda} = \sum_{m=1}^{q} \tilde{E}_m(\tilde{L})\,\vartheta^m \wedge \omega,$

- $$\boxed{\tilde{E}_m = \sum_{l=1}^{\tilde{q}} \mathcal{A}^{*l}{}_m \mathcal{E}_l - \sum_{i,j=1}^{p} \mathcal{B}^{*j}{}_{im} \mathcal{H}^i_j,}$$

- $d_{\tilde{V}}(\kappa^l) = \sum_{m=1}^{q} \mathcal{A}^l_m(\vartheta^m)$ and $d_{\tilde{V}}(\omega^j) = \sum_{i=1}^{p} \sum_{m=1}^{q} \mathcal{B}^j_{im}(\vartheta^m) \wedge \omega^i$

- $\mathcal{E}_l = \sum_{\hat{J}} \mathcal{D}^\dagger_{\hat{J}} \circ \dfrac{\partial}{\partial \kappa^m_{\hat{J}}}, \qquad \mathcal{H}^i_j = -\delta^i_j + \sum_{l=1}^{\tilde{q}} \sum_{\hat{J}\hat{K}} \kappa^l_{\widehat{Jj}}\, \mathcal{D}^\dagger_K \circ \dfrac{\partial}{\partial \kappa^l_{\widehat{JiK}}}.$

Noether Correspondence

$$\{\text{generalized symmetries of } \boxed{\lambda = L(\mathbf{x}, \mathbf{u}^{(n)})\, d\mathbf{x}}\} \,/\, \{\text{trivial symmetries}\}$$

$$\Updownarrow$$

$$\{\text{conservation laws of } \boxed{E(L) = 0}\} \,/\, \{\text{trivial conservation laws}\}$$

Noether Correspondence

$\{$generalized symmetries of $\boxed{\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x}}\}$ / $\{$trivial symmetries$\}$

$$\Updownarrow$$

$\{$conservation laws of $\boxed{E(L) = 0}\}$ / $\{$trivial conservation laws$\}$

$\mathbf{v} = \sum_{j=1}^{q} Q^j(\mathbf{x}, \mathbf{u}^{(k)}) \frac{\partial}{\partial u^j}$ is generalized variational symmetry if

$$\exists A = (A_1, \ldots, A_p), \quad \mathbf{v}^{\infty}(L) = \mathrm{Div} A.$$

$$\Updownarrow$$

$P = (P_1, \ldots, P_p)$ is a conservation law if $\mathrm{Div} P \equiv 0 \bmod E(L)$

In terms of differential forms

$\mathbf{v}$ is a gen. var. symmetry of $\lambda = L(\mathbf{x}, \mathbf{u}^{(n)})d\mathbf{x}$ if $\exists \alpha = \sum_{i=1}^{p} A_i \, d\widehat{\mathbf{x}}^i \in \Lambda^{p-1,0}$ s.t. $\boxed{\mathbf{v}^\infty(\lambda) = d_H(\alpha)}$ (equivalently $\mathbf{v}^\infty \lrcorner d_V \lambda = d_H(\alpha)$).

$$\Updownarrow$$

$\pi = \sum_{i=1}^{p} P_i \, d\widehat{\mathbf{x}}^i \in \Lambda^{p-1,0}$ is a conservation law of $E(L) = 0$ if $\boxed{d_H \pi = 0 \text{ mod } \{E(L)\}}$.

Noether correspondence: $\boxed{\pi = \mathbf{v}^\infty \lrcorner \nu + \alpha}$, where

$$d_V \lambda - \partial_V \lambda = d_H \nu, \quad \nu \in \Lambda^{p-1,1}$$

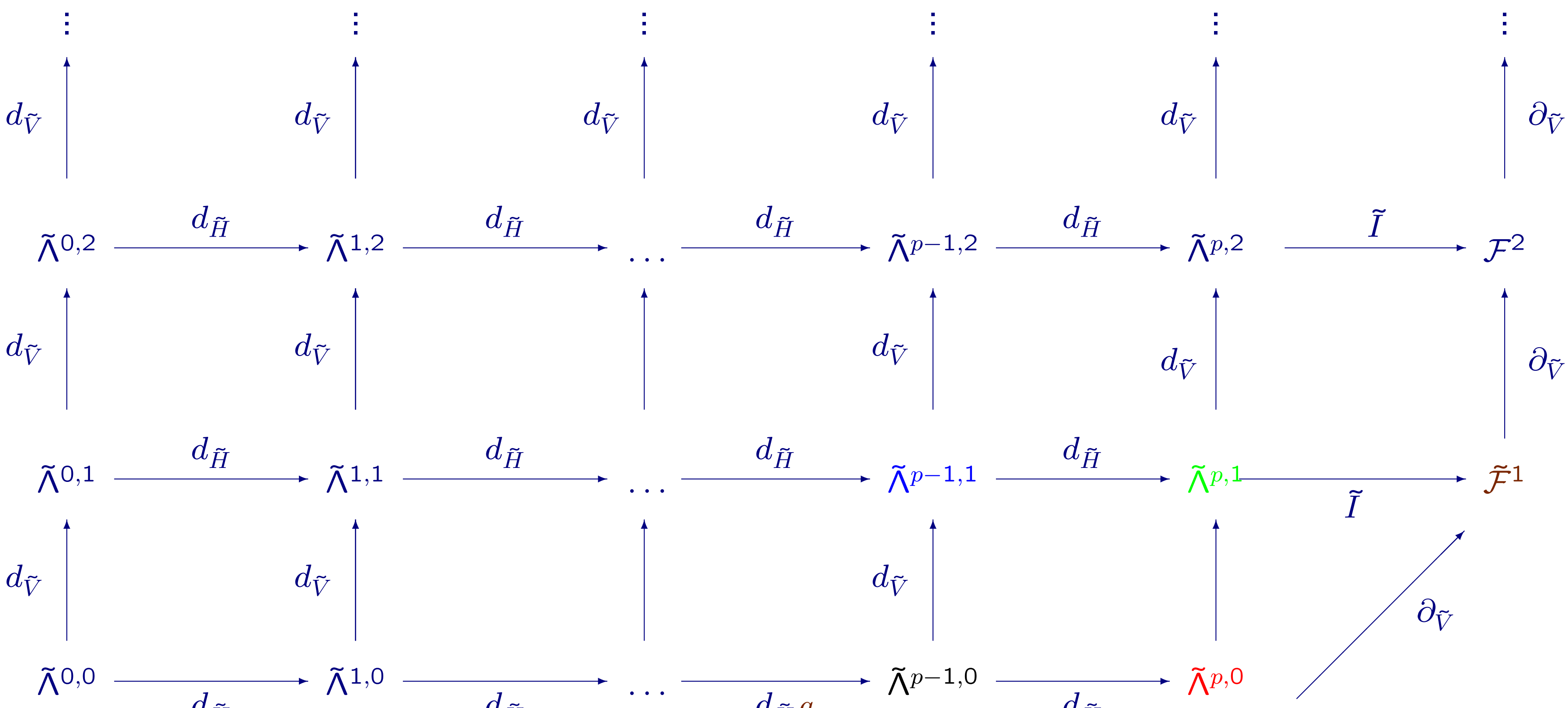
G-invariant Noether correspondence for G-symmetric variational problems

G-invariant generalized variational symmetry:

$$\downarrow (IK) \quad \uparrow ?$$

G-invariant conservation law:

Invariant Noether Correspondence



- $\lambda = \tilde L\,\omega \in \tilde\Lambda^{p,0}$ - Lagrangian; $\quad \partial_{\tilde V}\lambda = \sum_{m=1}^{q} \tilde E_m(\tilde L)\,\vartheta^m \wedge \omega$, ($\tilde E_m(\tilde L) = 0$ are E.-L. eq.).
- $d_{\tilde V}\lambda - \partial_{\tilde V}\lambda = d_{\tilde H}\nu, \quad \nu \in \Lambda^{p-1,1}$
- v.-f. $\mathbf{v}$ is an infinitesimal variational symmetry if $\exists \alpha \in \Lambda^{p-1,0}$ s.t. $\boxed{\mathbf{v}^\infty(\lambda) = d_{\tilde H}(\alpha)}$.
- Noether correspondence: $\boxed{\pi = \mathbf{v}^\infty \lrcorner \nu + \alpha}$ is a conservation law: $\boxed{d_{\tilde H}\pi = 0 \text{ mod } \{\tilde E_m(\tilde L)\}}$.

$SE(2)$-Invariant Example (Elastica)

- $\boxed{\lambda = \frac{1}{2}\kappa^2\omega}$, where $\quad \omega = ds + \frac{u_1}{\sqrt{1+u_x^2}}\theta$.

- $d_{\tilde{V}}\lambda = (\kappa_{ss} + \frac{1}{2}\kappa^3)\,\vartheta \wedge \omega \; + d_{\tilde{H}}\nu$, where $\boxed{\nu = \kappa_s\vartheta_1 - \kappa\vartheta_2}$.

- Euler-Lagrange equation: $\boxed{\tilde{E}(\tilde{L}) = \kappa_{ss} + \frac{1}{2}\kappa^3 = 0.}$

- Invariant evolutionary vector field: $\mathbf{v} = \psi(\kappa, \kappa_s)\,\mathcal{V}$, where
  $\mathcal{V} = -\frac{u_x}{\sqrt{1+u_x^2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{1+u_x^2}}\frac{\partial}{\partial u}$

- Symmetry condition $\boxed{\exists\alpha \quad \mathbf{v}^\infty(\lambda) = d_{\tilde{H}}\alpha} \Longrightarrow \psi = \kappa_s\, f(\kappa^4 + 4\kappa_s^2)$.

  Take $\mathbf{v} = \kappa_s\,\mathcal{V}$, then $\mathbf{v}^\infty(\lambda) = d_{\tilde{H}}\alpha$, where $\boxed{\alpha = \frac{1}{2}\,\kappa\kappa_{ss} + \frac{1}{8}\kappa^4 - \frac{1}{2}\kappa_s^2.}$

- Conservation laws: $\boxed{\pi = \mathbf{v}^\infty \lrcorner \nu + \alpha = \frac{1}{2}\kappa_s^2 + \frac{1}{8}\kappa^4}$

- Check: $\frac{d}{ds}\pi = \kappa_s(\kappa_{ss} + \frac{1}{2}\kappa^3) = \kappa_s\tilde{E}(\tilde{L}) \equiv 0 \bmod \tilde{E}(\tilde{L})$.

  (Recall $\tilde{E}(\tilde{L}) = \kappa_{ss} + \frac{1}{2}\kappa^3$).

$SA(2)$-Invariant Example

- $\boxed{\lambda = \mu\,\omega}$, where $\quad \omega = u_2^{1/3}\,dx + \frac{u_3}{3\,u_2^{5/3}}\theta.$

- $\mathcal{D} = \frac{1}{{u_2}^{1/3}}\frac{d}{dx}, \quad \mu_1 = \mathcal{D}\mu, \ldots, \mu_i = \mathcal{D}\mu_{i-1}.$

- $d_{\tilde{V}}\lambda = (\frac{2}{3}\mu_2 + \frac{2}{9}\mu^2)\,\vartheta \wedge \omega \; + d_{\tilde{H}}\nu$, where $\boxed{\nu = \frac{2}{3}\vartheta_1 - \frac{2}{3}\vartheta_2 - \vartheta_4}$.

- Euler-Lagrange equation: $\boxed{\tilde{E}(\tilde{L}) = -(\frac{2}{3}\mu_2 + \frac{2}{9}\mu^2) = 0.}$

- Invariant evolutionary vector field: $\mathbf{v} = \psi(\mu, \mu_1)\,\mathcal{V}$

- Symmetry condition $\boxed{\exists\alpha \quad \mathbf{v}^\infty(\lambda) = d_{\tilde{H}}\alpha} \Longrightarrow \psi = \mu_1\, f(\frac{2}{9}\mu^3 + \mu_1^2).$

  Take $\mathbf{v} = \mu_1\,\mathcal{V}$, then $\mathbf{v}^\infty(\lambda) = d_{\tilde{H}}\alpha$, where $\boxed{\alpha = \mu_4 + 2\mu\mu_2 + \frac{2}{27}\mu^3.}$

- Conservation laws: $\boxed{\pi = \mathbf{v}^\infty \lrcorner \nu + \alpha = \frac{2}{27}\mu^3 + \frac{1}{3}\mu_1^2}$

- Check: $\mathcal{D}\pi = \mu_1(\frac{2}{9}\mu^2 + \frac{2}{3}\mu_2) = \mu_1\tilde{E}(\tilde{L}) \equiv 0$ mod $\tilde{E}(\tilde{L})$.

  (Recall $\tilde{E}(\tilde{L}) = (\frac{2}{3}\mu_2 + \frac{2}{9}\mu^2)$).

Other work on Noether Correspondence and Moving Frames

- Gonçalves and Mansfield, 2011, 2013 consider a group G of point symmetries (not generalized symmetries) and use moving frame method to express conservation laws (not necessarily G-invariant) that correspond to G.

- We reduce a variational problem by its group G of point symmetries and consider G-invariant generalized symmetries of the reduced problem. This leads to invariant conservation laws.

Implemented for an invariant bicomplex in the iVB package

- $d_{\tilde{H}}$, $d_{\tilde{V}}$, $\tilde{I}$, $\partial_{\tilde{V}}$.

- Prolongation of the vector fields

- Exactness of interior rows: given $d_{\tilde{H}}(\omega) = 0$ where $\omega \in \tilde{\Lambda}^{t,s}$, $s > 0$, compute η s. t. $d_{\tilde{H}}(\eta) = \omega$.

Thank you!