Gaps, Symmetry, Integrability

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based on the joint work with Dennis The

The gap problem

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Example (Riemannian geometry in dim $= n$)

n	*max*	*submax*
2	3	1
3	6	4
4	10	8
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2}+1$

Darboux, Koenigs: $n = 2$ case

Wang, Egorov: $n \geq 3$ case

For other signatures the result is the same, except the 4D case

Parabolic geometry

We consider the gap problem in the class of parabolic geometries.

Parabolic geometry: Cartan geometry $(\mathcal{G} \to M, \omega)$ modelled on $(G \to G/P, \omega_{MC})$, where G is ss Lie grp, P is parabolic subgrp.

Examples

Model G/P	Underlying (curved) geometry
$SO(p+1, q+1)/P_1$	sign (p, q) conformal structure
$SL_{m+2}/P_{1,2}$	2nd ord ODE system in m dep vars
SL_{m+1}/P_1	projective structure in dim $= m$
G_2/P_1	$(2, 3, 5)$-distributions
$SL_{m+1}/P_{1,m}$	Lagrangian contact structures
$Sp_{2m}/P_{1,2}$	Contact path geometry
$SO(m, m+1)/P_m$	Generic $(m, \binom{m+1}{2})$ distributions

Known gap results for parabolic geometries

Geometry	*Max*	*Submax*	*Citation*
scalar 2nd order ODE mod point	8	3	Tresse (1896)
projective str 2D	8	3	Tresse (1896)
$(2,3,5)$-distributions	14	7	Cartan (1910)
projective str dim $= n+1$, $n \geq 2$	n^2+4n+3	n^2+4	Egorov (1951)
scalar 3rd order ODE mod contact	10	5	Wafo Soh, Qu Mahomed (2002)
conformal $(2,2)$ str	15	9	Kruglikov (2012)
pair of 2nd order ODE	15	9	Casey, Dunajski, Tod (2012)

Main results of Kruglikov & The (2012)

If the geometry $(\mathcal{G}, \omega)$ is flat $\kappa_H = 0$, then its (local) symmetry algebra has dimension $\dim G$. Let $\mathfrak{S}$ be the maximal dimension of the symmetry algebra $\mathcal{S}$ if M contains at least one non-flat point. Prev estimates of $\mathfrak{S}$: Čap–Neusser (2009), Kruglikov (2011)

Problem: Compute the number $\mathfrak{S}$

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Prev estimates of $\mathfrak{S}$: Čap–Neusser (2009), Kruglikov (2011)

Problem: Compute the number $\mathfrak{S}$

- For any complex or real regular, normal G/P geometry we give a universal upper bound $\mathfrak{S} \leq \mathfrak{U}$, where $\mathfrak{U}$ is algebraically determined via a Dynkin diagram recipe.
- In complex or split-real cases, we establish models with $\dim(\mathcal{S}) = \mathfrak{U}$ in almost all cases. Thus, $\mathfrak{S} = \mathfrak{U}$ almost always, exceptions are classified and investigated.
- Moreover we prove local homogeneity of all submaximally symmetric models near non-flat regular points.

Sample of new results on submaximal symmetry dimension

Geometry	Max	Submax
Sign (p,q) conf geom $n=p+q,\ \ p,q\geq 2$	$\binom{n+2}{2}$	$\binom{n-1}{2}+6$
Systems 2nd ord ODE in $m\geq 2$ dep vars	$(m+2)^2-1$	m^2+5
Generic m-distributions on $\binom{m+1}{2}$-dim manifolds	$\binom{2m+1}{2}$	$\begin{cases}\frac{m(3m-7)}{2}+10, & m\geq 4;\\ 11, & m=3\end{cases}$
Lagrangian contact str	m^2+2m	$(m-1)^2+4,\ m\geq 3$
Contact projective str	$m(2m+1)$	$\begin{cases}2m^2-5m+8, & m\geq 3;\\ 5, & m=2\end{cases}$
Contact path geometries	$m(2m+1)$	$2m^2-5m+9,\ m\geq 3$
Exotic parabolic contact structure of type E_8/P_8	248	147

Tanaka theory in a nutshell

Input: Distribution $\Delta \subset TM$ (possibly with structure on it) with the weak derived flag $\Delta^{-(i+1)} = [\Delta, \Delta^{-i}]$.

- filtration $\Delta = \Delta^{-1} \subset \Delta^{-2} \subset \cdots \subset \Delta^{-\nu} = TM$, ν - depth
- GNLA $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \ldots \oplus \mathfrak{g}_{-\nu}$, $\quad \mathfrak{g}_i = \Delta^i / \Delta^{i+1}$
- Graded frame bundle: $\mathcal{G}_0 \to M$ with str. grp. $G_0 \subset \mathrm{Aut}_{gr}(\mathfrak{m})$.
- Tower of bundles: ... $\to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0 \to M$. If finite, then

Output: Cartan geometry $(\mathcal{G} \to M, \omega)$ of some type (G, H).

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Tanaka's algebraic prolongation: $\exists!$ GLA $\mathfrak{g} = pr(\mathfrak{m}, \mathfrak{g}_0)$ s.t.

1. $\mathfrak{g}_{\leq 0} = \mathfrak{m} \oplus \mathfrak{g}_0$.
2. If $X \in \mathfrak{g}_+$ s.t. $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$.
3. $\mathfrak{g}$ is the maximal GLA satisfying the above properties.

Tanaka's prolongation of a subspace $\mathfrak{a}_0 \subset \mathfrak{g}_0$

Lemma

If $\mathfrak{a}_0 \subset \mathfrak{g}_0$*, then* $\mathfrak{a} = pr(\mathfrak{m}, \mathfrak{a}_0) \hookrightarrow \mathfrak{g} = pr(\mathfrak{m}, \mathfrak{g}_0)$ *is given by*

$$\mathfrak{a} = \mathfrak{m} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \dots, \text{ where } \mathfrak{a}_k = \{X \in \mathfrak{g}_k : \mathrm{ad}^k_{\mathfrak{g}_{-1}}(X) \subset \mathfrak{a}_0\}.$$

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Let $\mathfrak{p} \subset \mathfrak{g}$ be parabolic, so $\mathfrak{g} = \overbrace{\mathfrak{g}_{-\nu} \oplus \dots}^{\mathfrak{m}} \oplus \overbrace{\mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_\nu}^{\mathfrak{p}}$.

Theorem (Yamaguchi, 1993)

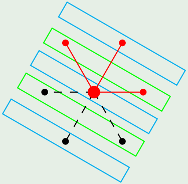
If $\mathfrak{g}$ is semisimple, $\mathfrak{p} \subset \mathfrak{g}$ is parabolic, then $pr(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{g}$ except for projective (SL_n/P_1) and contact projective (Sp_{2n}/P_1) str.

Example (2nd order ODE $y'' = f(x, y, y')$ mod point transf.)

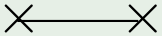
$M : (x, y, p)$, $\Delta = \{\partial_p\} \oplus \{\partial_x + p\partial_y + f(x, y, p)\partial_p\}$.
$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$, where $\mathfrak{g}_{-1} = \mathfrak{g}'_{-1} \oplus \mathfrak{g}''_{-1}$. Also, $\mathfrak{g}_0 \cong \mathbb{R} \oplus \mathbb{R}$.

Same as SL_3/B data:

$$\mathfrak{sl}_3 = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \Leftrightarrow$$



$\Leftrightarrow$



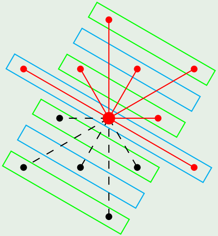
$$\mathfrak{sl}_3 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}^{\mathfrak{b}=\mathfrak{p}_{1,2}} . \mathfrak{g}_{-1} = \mathfrak{g}'_{-1} \oplus \mathfrak{g}''_{-1}, \quad \mathfrak{g}_0 \cong \mathbb{R} \oplus \mathbb{R}$$

Yamaguchi: $pr(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{sl}_3$.
Any 2nd order ODE = (SL_3, B)-type geom.

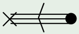
Example ((2, 3, 5)-distributions)

Any such Δ can be described as Monge eqn $z' = f(x, z, y, y', y'')$.
$M : (x, z, y, p, q)$, $\Delta = \{\partial_q, \partial_x + p\partial_y + q\partial_p + f\partial_z\}$, $f_{qq} \neq 0$.
$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}$ with dims $(2, 1, 2)$, and $\mathfrak{g}_0 = \mathfrak{gl}_2$.

Same as
G_2/P_1 data:



$\Leftrightarrow$



$$Lie(G_2) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3}^{\mathfrak{p}_1}$$

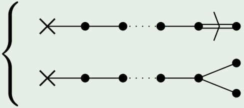
Yamaguchi $pr(\mathfrak{m}, \mathfrak{g}_0) = Lie(G_2)$.
Any (2, 3, 5)-dist. = (G_2, P_1)-type geom.

Example (Conformal geometry)

Let $(M, [\mu])$ be sig. (p, q) conformal mfld, $n = p + q$. Here, $\Delta = TM$, $\mathfrak{m} = \mathfrak{g}_{-1}$, and $\mathfrak{g}_0 = \mathfrak{co}(\mathfrak{g}_{-1})$.

Same as $SO_{p+1,q+1}/P_1$ data: if $g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then

$$\mathfrak{so}_{p+1,q+1} = \begin{pmatrix} 0 & 1 & \cdot \\ -1 & 0 & 1 \\ \cdot & -1 & 0 \end{pmatrix} \quad \Leftrightarrow$$



(n odd);

(n even).

$$\mathfrak{so}_{p+1,q+1} = \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{p}_1}$$

Yamaguchi $pr(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{so}_{p+1,q+1}$.

Any conformal geometry = $(SO_{p+1,q+1}, P_1)$-type geom.

Harmonic curvature

Curvature: $K = d\omega + \frac{1}{2}[\omega,\omega] \quad \Leftrightarrow \quad \kappa : \mathcal{G} \to \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$.

Kostant: $$\bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g} = \overbrace{\mathrm{im}(\partial^*) \oplus \underbrace{\mathrm{ker}(\Box)}_{}}^{\mathrm{ker}(\partial^*)}\!\!\!\!\underbrace{\phantom{\mathrm{ker}(\Box)}\oplus \mathrm{im}(\partial)}_{\mathrm{ker}(\partial)} \text{ (as } \mathfrak{g}_0\text{-modules)}$$

Normality: $\partial^*\kappa = 0$. A simpler object is harmonic curvature κ_H:

- $\kappa_H : \mathcal{G}_0 \to H^2_+(\mathfrak{m},\mathfrak{g})$ (G_0-equivariant)
- $(\mathcal{G} \to M, \omega)$ is locally flat iff $\kappa_H = 0$.

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$$\bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g} = \overbrace{\mathrm{im}(\partial^*) \oplus \mathrm{ker}(\Box)}^{\mathrm{ker}(\partial^*)} \oplus \mathrm{im}(\partial) = \mathrm{im}(\partial^*) \oplus \underbrace{\mathrm{ker}(\Box) \oplus \mathrm{im}(\partial)}_{\mathrm{ker}(\partial)}$$

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Examples

Geometry	Curvature κ_H
conformal	Weyl ($n \geq 4$) or Cotton ($n = 3$)
(2,3,5)-distributions	Cartan's binary quartic
2nd order ODE	Tresse invariants (I, H)

Kostant's version of Bott–Borel–Weil thm (1961)

Input: G/P & $\mathfrak{p}$-rep $\mathbb{V}$. Output: $H^*(\mathfrak{m}, \mathbb{V})$ as a $\mathfrak{g}_0$-module.

Baston–Eastwood (1989): Expressed $H^2_+(\mathfrak{m}, \mathfrak{g})$ (the space where κ_H lives) in terms of weights and marked Dynkin diagrams.

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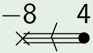
Example (G_2/P_1 geometry $\Leftrightarrow$ $(2,3,5)$-distributions)

Here, $\mathfrak{g}_0 = \mathcal{Z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss} = \mathbb{C} \oplus \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{gl}_2(\mathbb{C})$. The output of Kostant's BBW thm is:

$H^2(\mathfrak{m}, \mathfrak{g}) =$ 

$= \bigodot^4(\mathfrak{g}_1) = \bigodot^4(\mathbb{R}^2)^*.$

c.f. Cartan's 5-variables paper (1910) via method of equivalence.

General dim bound for regular normal parabolic geometries

$$\phi \in H^2_+,\ \mathfrak{a}_0^\phi = \mathfrak{ann}(\phi) \subseteq \mathfrak{g}_0,\ \mathfrak{a}^\phi = pr(\mathfrak{g}_-, \mathfrak{a}_0^\phi) = \mathfrak{g}_- \oplus \mathfrak{a}_0^\phi \oplus \mathfrak{a}_1^\phi \oplus \dots$$

Theorem

For G/P geom: $\quad \dim(\mathfrak{inf}(\mathcal{G},\omega)) \leq \inf_{x\in M} \dim(\mathfrak{a}^{\kappa_H(x)}).$

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Theorem

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To maximize the r.h.s. decompose into $\mathfrak{g}_0$-irreps: $H^2_+ = \bigoplus_i \mathbb{V}_i$, $\phi = \sum_i \phi_i$. Let $v_i \in \mathbb{V}_i$ be the lowest weight vectors.

Proposition (Complex case)

$$\max_{0 \neq \phi \in H^2_+} \dim(\mathfrak{a}_k^\phi) = \max_i \dim(\mathfrak{a}_k^{v_i}), \qquad \forall k \geq 0.$$

This implies the universal upper bound $\mathfrak{U} = \max_i \dim(\mathfrak{a}^{v_i})$

General dim bound for regular normal parabolic geometries

Theorem

We have $\mathfrak{S} \leq \mathfrak{U}$ *and the bound is sharp in almost all cases.*

More precisely we have: $\mathfrak{S} = \mathfrak{U}$ except in the following cases.

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More precisely we have: $\mathfrak{S} = \mathfrak{U}$ except in the following cases.

List of exceptions:

- A_2/P_1 (2D projective structure). Here $\mathfrak{S} = 3 < 4 = \mathfrak{U}$.
- $A_2/P_{1,2}$ (scalar 2nd ord ODE mod point $\equiv$ 3D CR str). Here $\mathfrak{S} = 3 < 4 = \mathfrak{U}$.
- B_2/P_1 (3D conformal Riemannian/Lorenzian structures). Here $\mathfrak{S} = 4 < 5 = \mathfrak{U}$.
- $G/P = A_1/P_1 \times G'/P'$ (semi-simple case with split curvature). Here $\mathfrak{U} - 1 \leq \mathfrak{S} \leq \mathfrak{U}$.

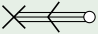
Dynkin diagram recipes - 1

1. From $\mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$, we have $\mathfrak{g}_0 = \mathcal{Z}(\mathfrak{g}_0) \oplus (\mathfrak{g}_0)_{ss}$ with

$$\begin{cases} \dim(\mathcal{Z}(\mathfrak{g}_0)) = \#\text{ crosses;} \\ (\mathfrak{g}_0)_{ss}\text{ D.D.} \to \text{remove crosses.} \end{cases}$$

Since $\dim(\mathfrak{g}_-) = \dim(\mathfrak{g}_+)$, get $n = \dim(\mathfrak{g}/\mathfrak{p})$ and $\dim(\mathfrak{p})$.

Example (G_2/P_1)

 , $\dim(\mathfrak{g}_0) = 4$, $n = 5$.

Dynkin diagram recipes - 2

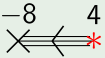
Let $\mathbb{V} \subset H^2_+$ be a $\mathfrak{g}_0$-irrep.

2. $\dim(\mathfrak{ann}(\phi))$ $(0 \neq \phi \in \mathbb{V})$ is max on l.w.vector $\phi = \phi_0 \in \mathbb{V}$, $\mathfrak{q} := \{X \in (\mathfrak{g}_0)_{ss} \mid X \cdot \phi_0 = \lambda \phi_0\}$ is parabolic, and

$$\boxed{\dim(\mathfrak{ann}(\phi_0)) = (\#crosses) - 1 + \dim(\mathfrak{q})}.$$

D.D. notation: If $\neq 0$ on uncrossed node, put $*$.

Example (G_2/P_1)

$H^2_+ =$ 

, $\dim(\mathfrak{ann}(\phi_0)) = 2$.

Dynkin diagram recipes - 3

Let $\mathbb{V} \subset H^2_+$ be a $\mathfrak{g}_0$-irrep.

Lemma

$\dim(\mathfrak{a}^\phi_+)$ $(0 \neq \phi \in \mathbb{V})$ *is max on l.w.vector* $\phi = \phi_0 \in \mathbb{V}$.

Let $\mathbb{V} \subset H^2_+$ be a $\mathfrak{g}_0$-irrep.

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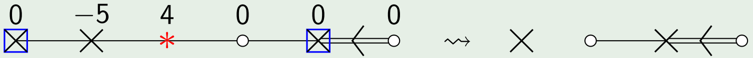
D.D. notation: If 0 over $\times$ $\rightsquigarrow$ put $\square$.

3. Remove all $*$ and $\times$, except $\square$, and the adjacent edges. Remove connected components without $\square$. Obtain $(\bar{\mathfrak{g}}, \bar{\mathfrak{p}})$.

Example ($C_6/P_{1,2,5} \rightsquigarrow A_1/P_1 \times C_3/P_2$: $M^{32} = \tilde{G}^{51}/H^{19}$)



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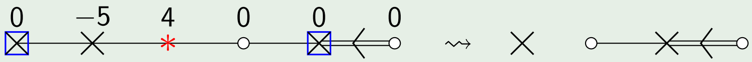
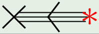


Proposition

No $\square \Leftrightarrow \dim(\mathfrak{a}^{\phi_0}_+) = 0$. *Otherwise* $\dim(\mathfrak{a}^{\phi_0}_+) = \dim(\bar{\mathfrak{g}}/\bar{\mathfrak{p}})$.

Examples of computations

Example

G/P	H^2_+ components	n	$\dim(\mathfrak{a}_0^{\phi_0})$	$\dim(\mathfrak{a}_+^{\phi_0})$	$\dim(\mathfrak{a}^{\phi_0})$
G_2/P_1	-8 4 	5	2	0	7

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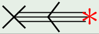
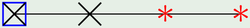
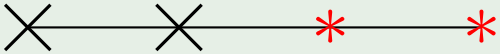
G/P	H^2_+ components	n	$\dim(\mathfrak{a}_0^{\phi_0})$	$\dim(\mathfrak{a}_+^{\phi_0})$	$\dim(\mathfrak{a}^{\phi_0})$
G_2/P_1	-8 4 	5	2	0	7
$A_4/P_{1,2}$	0 -4 3 1 	7	6	1	14
	-4 1 1 1 	7	6	0	13

Examples of computations

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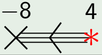
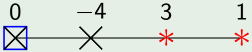
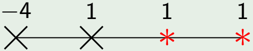
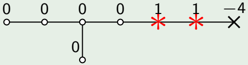
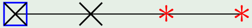
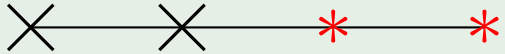
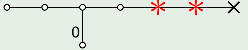
G/P	H^2_+ components	n	$\dim(\mathfrak{a}_0^{\phi_0})$	$\dim(\mathfrak{a}_+^{\phi_0})$	$\dim(\mathfrak{a}^{\phi_0})$
G_2/P_1		5	2	0	7
$A_4/P_{1,2}$		7	6	1	14
		7	6	0	13
E_8/P_8		57	90	0	147

Example

G/P	H^2_+ components	n	$\dim(\mathfrak{a}_0^{\phi_0})$	$\dim(\mathfrak{a}_+^{\phi_0})$	$\dim(\mathfrak{a}^{\phi_0})$
G_2/P_1		5	2	0	7
$A_4/P_{1,2}$		7	6	1	14
		7	6	0	13
E_8/P_8		57	90	0	147

Proposition (Maximal parabolics)

Single cross $\Rightarrow$ *no* $\square$, *so* $\mathfrak{a}_+^{\phi_0} = 0$.

We classified all complex $(\mathfrak{g}, \mathfrak{p})$ with $\mathfrak{a}_+^{\phi_0} \neq 0$ with $\mathfrak{g}$ simple. This gives all complex nonflat geometries with higher order fixed points.

Ex of finer str's: 4D Lorentzian conformal geometry

$SO(2,4)/P_1$ geometry: $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{so}(1,3) = \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$,

$$H^2_+ \cong \bigodot{}^4 \mathbb{C}^2 \qquad \text{(as a } \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\text{-rep)}$$

and $Z \in \mathcal{Z}(\mathfrak{g}_0)$ acts with homogeneity $+2$. $\mathbb{C}$-basis of $\mathfrak{sl}(2,\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Petrov type	Normal form in $\bigodot^4(\mathfrak{g}_1)$	Annihilator $\mathfrak{h}_0$	$\dim(\mathfrak{h})$	sharp?
N	ξ^4	$X, iX, 2Z - H$	7	✓
III	$\xi^3\eta$	$Z - 2H$	5	×
D	$\xi^2\eta^2$	H, iH	6	✓
II	$\xi^2\eta(\xi-\eta)$	0	4	✓
I	$\xi\eta(\xi-\eta)(\xi-k\eta)$	0	4	✓

Thus, $\boxed{\textit{submax} \leq 7}$. Sharp for CKV's of the $(1,3)$ pp-wave:

$$ds^2 = dx^2 + dy^2 + 2dz\,dt + x^2 dt^2.$$

Further developments

- We proved (Kruglikov & The) recently: Every automorphism of a parabolic geometry is 2-jet determined; in non-flat regular points it is 1-jet determined.
- In several occasions we classified all sub-maximal models via deformations of the filtered Lie algebras of symmetries. The general question is however open.
- Non-split real parabolic geometries are still open. Recently Doubrov & The found the submaximal dimensions for Lorentzian conformal structures in dim ≥ 4 (for other signatures and in 3D this was done by Kruglikov & The).
- Some geometric structures that are specifications of parabolic geometries can be elaborated using our results. Recently Kruglikov & Matveev obtained submaximal dimensions for metric projective and metric affine structures.

Examples of submaximal symmetric models

General signature conformal str: The submaximal structure is unique and is given by the pp-wave metric

$$ds^2 = dx\,dy + dz\,dt + y^2 dt^2 + \epsilon_1 du_1^2 + \cdots + \epsilon_{n-4} du_{n-4}^2.$$

It is Einstein (Ricci-flat) in any dimension and self-dual in 4D. Its geodesic flow is integrable in both classical and quantum sense.

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(2,3,5)-distributions: The submaximal structures have 1D moduli. They are parametrized by the Monge underdetermined ODE $y' = (z'')^m$, $2m-1 \notin \{\pm 1/3, \pm 1, \pm 3\}$, and also a separate model $y' = \ln(z'')$. Deformations of these structures lead via Fefferman-Graham and Nurowski constructions to Ricci flat metrics with special holonomies (G_2, Heiseinberg).

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3rd ord ODE mod contact: Maximal structures $y''' = 0$ have 10D symm. Submaximal structures have 5D symm, and are linearizable (with constant coefficients). They are exactly solvable.

Scalar 2nd ord ODE mod point: Submaximal metrizable models here represent super-integrable geodesic flows. Non-metrizable equations are also integrable (solvable in quadratures).

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Systems of 2nd ord ODE: The submaximal structure is given by

$$\ddot{x}_1 = 0, \quad \dots, \quad \ddot{x}_{n-1} = 0, \quad \ddot{x}_n = \dot{x}_1{}^3.$$

It is solvable via simple quadrature, and is an integrable extension of the flat ODE system in $(n-1)$ dim (uncoupled harmonic oscillators). Moreover for this system Fels' T-torsion vanishes, and so it determines an integrable Segré structure.

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Projective structures: Every projective structure can be written via its equation of geodesics (defined up to projective reparametrization). The submaximal model then writes

$$\ddot{x}_1 = x_1\dot{x}_1{}^2\dot{x}_2, \;\; \ddot{x}_2 = x_1\dot{x}_1\dot{x}_2{}^2, \;\; \ddot{x}_3 = x_1\dot{x}_1\dot{x}_2\dot{x}_3, \;\; \dots, \;\; \ddot{x}_n = x_1\dot{x}_1\dot{x}_2\dot{x}_n.$$

This system is solvable via quadrature. Its Fels' S-curvature is 0.

Nice properties of the submaximal symmetric structures should not be overestimated. Examples:

- submaximal projective structures are not metrizable,
- submax 2nd ord ODE systems are not projective connections.

Parabolic geometries with additional structures also have nice properties. Example:

- Both submaximal projective metric structures and submaximal affine metric structures have integrable geodesic flows.

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The gap problem is more complicated for general geometries. Already for vector distributions, the maximal and submaximal dimensions of the symmetry group often differ by 1. This absence of gap is related to the structure of the max symmetry groups.

Generalizations

Similar problem arises for infinite-dimensional pseudogroups acting on differential equations and soft geometric structures. Examples:

♢ Parabolic Monge-Ampére equations in 2D have the symmetry pseudogroup depending on at most 3 function of 3 arguments. In the case of elliptic/hyperbolic equations it reduces to 2 functions of 2 arguments. In higher dimensions non-degeneracy of the symbol also reduces the possible functional dimension.

♢ For the Cauchy-Riemann equation, describing pseudoholomorphic curves and submanifolds, the maximal functional dimension corresponds to the integrable almost complex structure. In the submaximal cases integrability is manifested by the existence of pseudoholomorphic foliations.

Integrable symplectic Monge-Ampére equations

In 4D such equations of Hirota type were classified up to $Sp(8)$ by Doubrov & Ferapontov. There are 5 non-linearizable PDEs, all kinds of the heavenly equations.

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An important fact is that all of them possess a huge algebra of symmetries – it is parametrized by 4 functions of 2 arguments: the symmetry pseudogroup consists of 4 copies of $SDiff(2)$ (joint work BK & Morozov). Moreover these compose into a graded group, exhausting all monoidal structures on the set of 4 elements, and the symmetry pseudogroup uniquely determines the corresponding integrable equation via differential invariants (following the general theory developed by BK & Lychagin).

Integrable dispersionless PDEs in 3D etc

The symbol of the formal linearization of a scalar PDE is an important geometric invariant reflecting the integrability properties.

For example, linearization of the 2nd order dispersionless PDE can be expressed as flatness (maximal symmetry) of the conformal metric that is the inverse of the symbol symmetric bivector. It is yet to interpret the submaximal property of the symbol.

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Integrability of the 2nd order dispersionless PDE is a more subtle property, but it can also be tested via the geometry of the formal linearization (joint project BK & Ferapontov). Namely (under some assumptions) integrability is equivalent to Einstein-Weyl property of the conformal structure of the inverse to linearization on the solution space. Similar results hold in 4D, where the Einstein-Weyl property is changed by the self-duality of the symbol.