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Multiple differentiation processes in differential geometry

Kirill Mackenzie

Sheffield, UK

Focused Research Workshop on Exterior Differential Systems and Lie Theory

Fields Institute
December 13, 2013

1. Introduction

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- Lie groupoids *(groupoïdes différentiables)*
- Jets
- multiple categories
- (and much else)

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There are double and multiple versions of this.

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Horizontal composition (when $v_1' = v_2$) has vertical sources and targets as follows :

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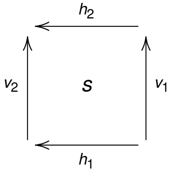

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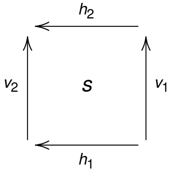

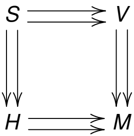

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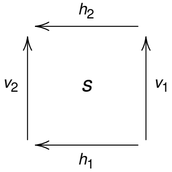

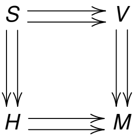

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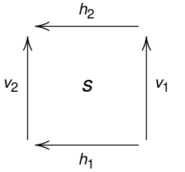

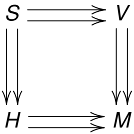

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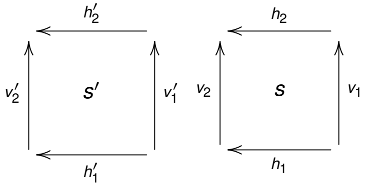

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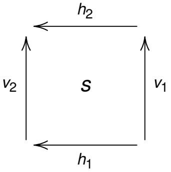

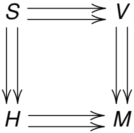

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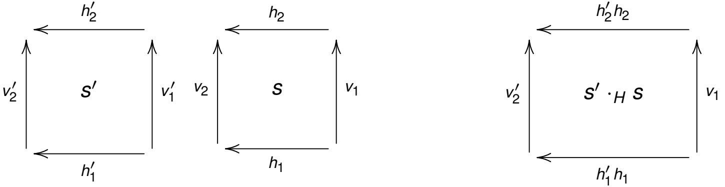

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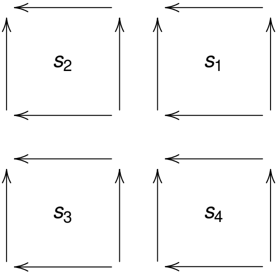

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composing each row horizontally and then the results vertically

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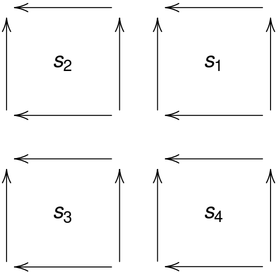

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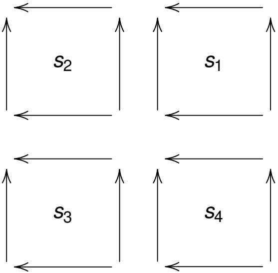

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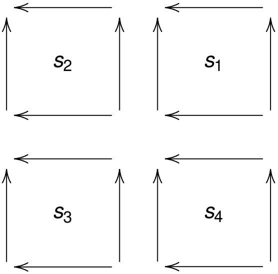

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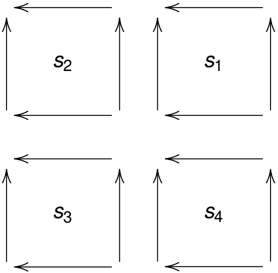

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5. Lie algebroids of a double Lie groupoid

Given a double Lie groupoid, one can take the Lie algebroid of either groupoid structure on S.

$$\begin{array}{ccc} S & \rightrightarrows & V \\ \downdownarrows & & \downdownarrows \\ H & \rightrightarrows & M \end{array}$$

Take the Lie algebroid of the vertical structure; the horizontal groupoid structure prolongs to the vertical Lie algebroid.

$$\begin{array}{ccc} A_V S & \rightrightarrows & AV \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array}$$

Take the Lie algebroid of the horizontal groupoid.

$$\begin{array}{ccc} A_H(A_V S) & \longrightarrow & AV \\ \downarrow & & \downarrow \\ AH & \longrightarrow & M \end{array}$$

$A_H(A_V S)$ is a Lie algebroid over base AV.

The vertical structure $A_H(A_V S) \to AH$ is at present just a vector bundle.

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The result is the *double Lie algebroid* of S.

7. Basic example

For $G \rightrightarrows M$ any Lie groupoid, take $S = G \times G$

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There is a canonical diffeomorphism $T(AG) \cong A(TG)$.

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Put $G = M \times M$. Then the preceding example is $S = M^4$ and the two forms of the double Lie algebroid are

$$\begin{array}{ccc} T(TM) & \xrightarrow{p_{TM}} & TM \\ {\scriptstyle T(p)}\downarrow & & \downarrow{\scriptstyle p} \\ TM & \xrightarrow[p]{} & M \end{array} \qquad\qquad \begin{array}{ccc} T(TM) & \xrightarrow{T(p)} & TM \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$$

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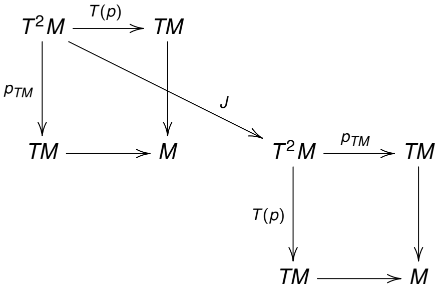

9. Local representation

Take $\xi \in T^2M$ with projections

$$\begin{array}{ccc} \xi & \xrightarrow{p_{TM}} & Y \\ {\scriptstyle T(p)}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & m \end{array}$$

If $X = 0$ then ξ is vertical and if $Y = 0$ then ξ is at a zero.

So if $X = Y = 0$ then ξ can be identified with an element Z of TM.

Represent elements of T^2M 'locally' as (X, Y, Z) where the Z is called a *core element*.

Write T^2M 'locally' as $TM * TM * TM$.

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10. Local representation, p2

More generally, for any vector bundle E on M, there is a double vector bundle

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ {\scriptstyle T(q)}\downarrow & & \downarrow{\scriptstyle q} \\ TM & \xrightarrow[p]{} & M \end{array}$$

Write elements as

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The e_2 is the core element.

Now dualize TE over E and we get

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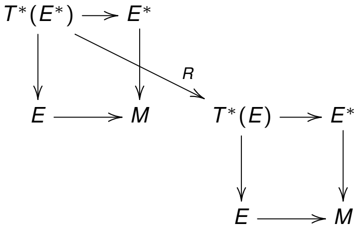
11. Canonical diffeomorphism R

For any vector bundle E there is an isomorphism of double vector bundles



Locally this is $(\varphi, e, \theta) \mapsto (e, \varphi, -\theta)$ where $\varphi \in E^*$, $e \in E$, $\theta \in T^*M$.

Apply this to $E = TM$ and we get $R\colon\ T^*(T^*M) \to T^*(TM)$,

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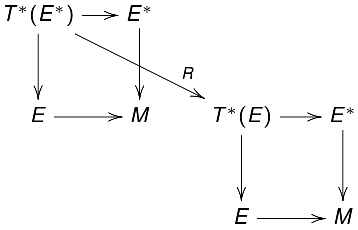

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12. Canonical diffeomorphism $\sharp$

The canonical symplectic structure $d\lambda$ on T^*M induces an isomorphism

$$\sharp\colon\ T^*(T^*M) \to T(T^*M).$$

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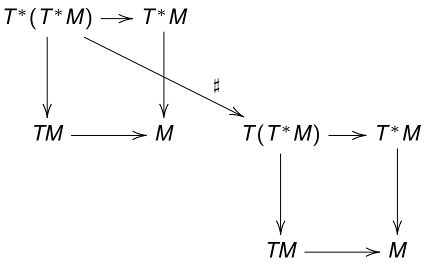

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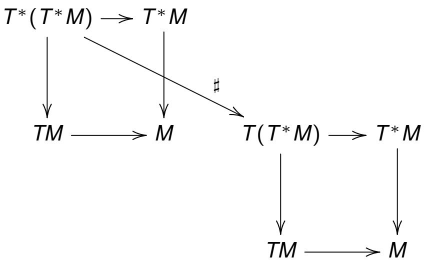

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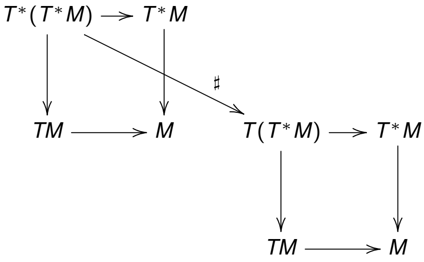

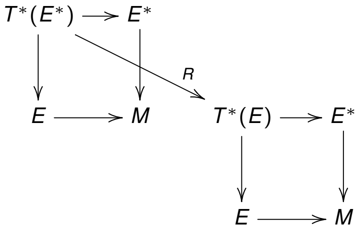

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\begin{array}{ccc}
T^*T^*M & \xrightarrow{\ R\ } & T^*TM \\
\big\downarrow \sharp & \nearrow \Theta & \\
TT^*M & &
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This all extends to double Lie groupoids. The question is, why do we want to ?

14. Double Lie groupoids again

Take the Lie algebroids of a double Lie groupoid S :

$$\begin{array}{ccc} A_V S & \rightrightarrows & AV \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array} \qquad \begin{array}{ccc} A_H S & \longrightarrow & V \\ \downarrow\downarrow & & \downarrow\downarrow \\ AH & \longrightarrow & M \end{array}$$

In each case take the dual. We get

$$\begin{array}{ccc} A^*_V S & \rightrightarrows & A^* K \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array} \qquad \begin{array}{ccc} A^*_H S & \longrightarrow & V \\ \downarrow\downarrow & & \downarrow\downarrow \\ A^* K & \longrightarrow & M \end{array}$$

The groupoid $K \rightrightarrows M$ here is the ‘core groupoid’ of S. The elements of K are the $s \in S$ for which both sources are identity elements.

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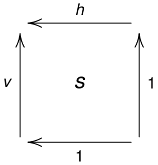

15. Theorem :

$A_V^* S \rightrightarrows A^* K$ and $A_H^* S \rightrightarrows A^* K$ are Poisson groupoids with respect to the Lie-Poisson structures, and are in duality as Poisson groupoids.

In particular, there is an isomorphism of Lie algebroids

$$\widetilde{\sharp}\colon\ A^*(A_V^* S) \to A(A_H^* S).$$

For $S = M^4$ this is $\sharp\colon\ T^*(T^*M) \to T(T^*M)$.

Further there is a commutative diagram.

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$A^*_V S \rightrightarrows A^* K$ and $A^*_H S \rightrightarrows A^* K$ are Poisson groupoids with respect to the Lie-Poisson structures, and are in duality as Poisson groupoids.

In particular, there is an isomorphism of Lie algebroids

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For $S = M^4$ this is $\sharp\colon\ T^*(T^*M) \to T(T^*M)$.

Further there is a commutative diagram.

$$\begin{array}{ccc} A^*(A^*_V S) & \xrightarrow{\ \widetilde{R}\ } & A^*(A_V S) \\ {\scriptstyle \widetilde{\sharp}}\downarrow & \nearrow {\scriptstyle \widetilde{\Theta}} & \\ A(A^*_H S) & & \end{array}$$

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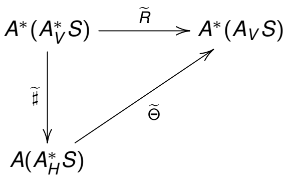

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16. Remark on Poisson group(oid)s

For G a Poisson Lie group:

$$\begin{array}{ccc} \Sigma & \rightrightarrows & G^* \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & \{\cdot\} \end{array} \quad \overset{\text{int}}{\Longleftarrow} \quad \begin{array}{ccc} T^*G & \rightrightarrows & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ G & \rightrightarrows & \{\cdot\} \end{array} \quad \overset{\text{diff}}{\Longrightarrow} \quad \begin{array}{ccc} \mathfrak{g} \times \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & \{\cdot\} \end{array}$$

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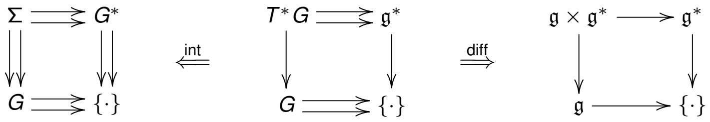

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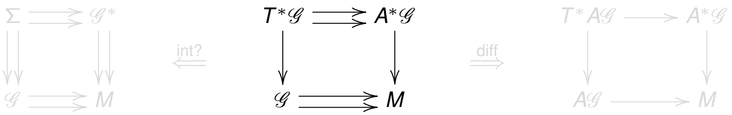

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17. *n*-fold Lie algebroids; super formulation (Th. Voronov)

A Q*-manifold* is a super vector bundle E on M with a homological vector field Q of weight 1. 'Homological' means $Q^2 = 0$.

Write $A = \Pi E$ for the parity reversed bundle.

Write i for the natural odd injection

$$i\colon\ \Gamma A \to \mathscr{X}(A),$$

Then Q defines a Lie algebroid structure on A with anchor

$$a(u)f := \big[[Q, i(u)], f\big]$$

and bracket

$$i([u,v]) := (-1)^u\big[[Q, i(u)], i(v)\big].$$

for $f \in C^\infty(M)$, and $u, v \in \Gamma A$. (Vaĭntrob.)

In local coordinates (x^a in the base, ξ^i in the parity-reversed fibres)

$$Q = \xi^i Q_i^a(x)\,\frac{\partial}{\partial x^a} + \frac{1}{2}\,\xi^i\xi^j Q_{ji}^k(x)\,\frac{\partial}{\partial \xi^k}\,.$$

Given a super double vector bundle, and writing D for the double-parity-reversed double vector bundle, two homological vector fields Q_1, Q_2 define a double Lie algebroid structure on D if

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This extends in a ready fashion to the n-fold case.

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17. n-fold Lie algebroids; super formulation (Th. Voronov)

A Q*-manifold* is a super vector bundle E on M with a homological vector field Q of weight 1. 'Homological' means $Q^2 = 0$.

Write $A = \Pi E$ for the parity reversed bundle.

Write i for the natural odd injection

$$i\colon \ \Gamma A \to \mathscr{X}(A),$$

Then Q defines a Lie algebroid structure on A with anchor

$$a(u)f := \big[[Q, i(u)], f\big]$$

and bracket

$$i([u, v]) := (-1)^u \big[[Q, i(u)], i(v)\big].$$

for $f \in C^\infty(M)$, and $u, v \in \Gamma A$. (Vaĭntrob.)

In local coordinates (x^a in the base, ξ^i in the parity-reversed fibres)

$$Q = \xi^i Q_i^a(x)\,\frac{\partial}{\partial x^a} + \frac{1}{2}\,\xi^i \xi^j Q_{ji}^k(x)\,\frac{\partial}{\partial \xi^k}\,.$$

Given a super double vector bundle, and writing D for the double-parity-reversed double vector bundle, two homological vector fields Q_1, Q_2 define a double Lie algebroid structure on D if

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18. A few references

For double Lie groupoids and double Lie algebroids see

• KM, Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids. *J. Reine Angew. Math.*, 658:193–245, 2011.

and earlier KM papers cited there.

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• The formulation of Lie algebroids in terms of Q-manifolds is from

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19. End frame