Combinatorial Algorithms to Solve Network Interdiction and Scheduling Problems with Multiple Parameters

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Sauder School of Business, UBC; U. Rome Tor Vergata; Aachen

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Outline

1. Network Interdiction
   - What is it?
   - Interdiction curves

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2. LP Duality
   - Dual of interdiction

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   - Algorithms
   - Discrete Newton
5. Multiple Parameters
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   - Scheduling problem
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- Finally, we have a budget $B \geq 0$ to spend on destroying arcs. Our objective is to spend at most B (maybe fractionally) in a way that minimizes the value of the residual flow.
  - In Min Cut we remove arcs until there is zero flow left, but here we remove only as much as we can under the budget.

Removing arcs greedily

- Thus if $B = 0$, then the interdiction value is cap_c^*, the ordinary min cut value; for $B \geq \mathrm{cap}_r^*$, the interdiction value is 0.

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  - Proof: again we could use a pairwise interchange argument.
- So let's get some idea of how much flow we can remove by destroying arcs from a fixed cut S.

The interdiction curve for a fixed cut S

Assume that we concentrate all our destruction on arcs of S.

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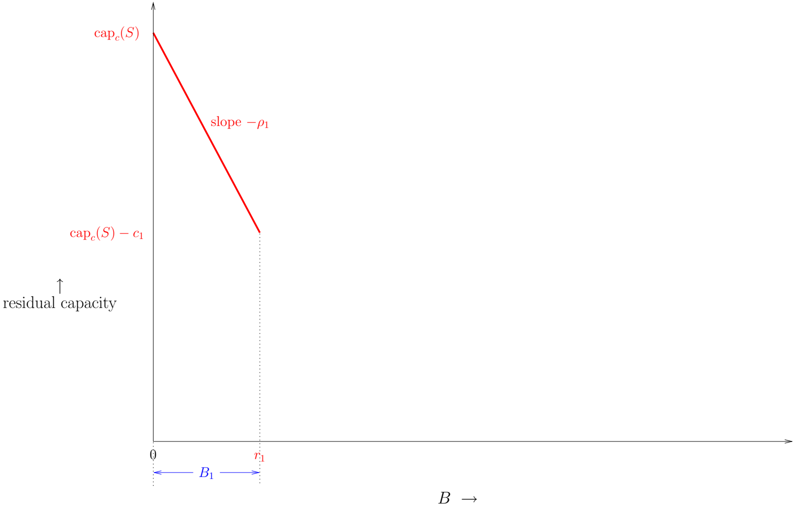

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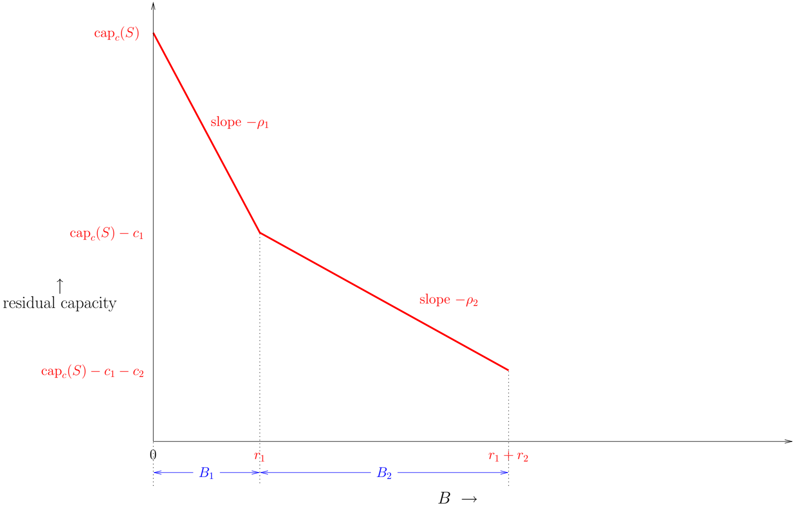

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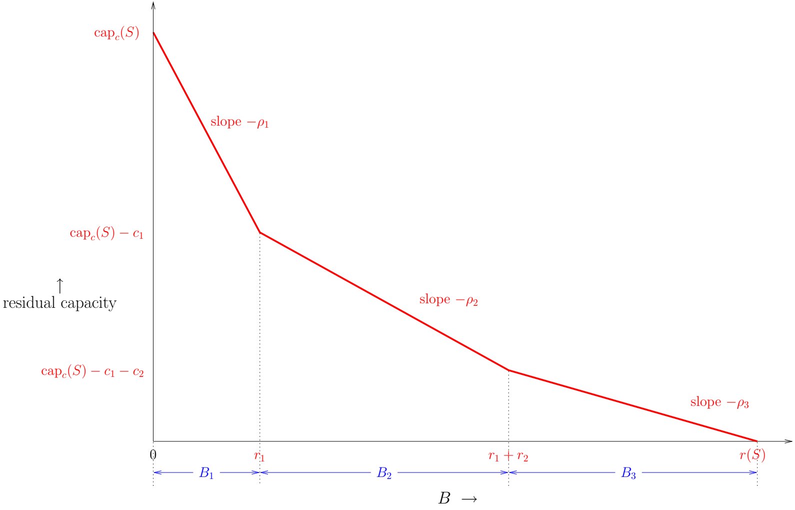
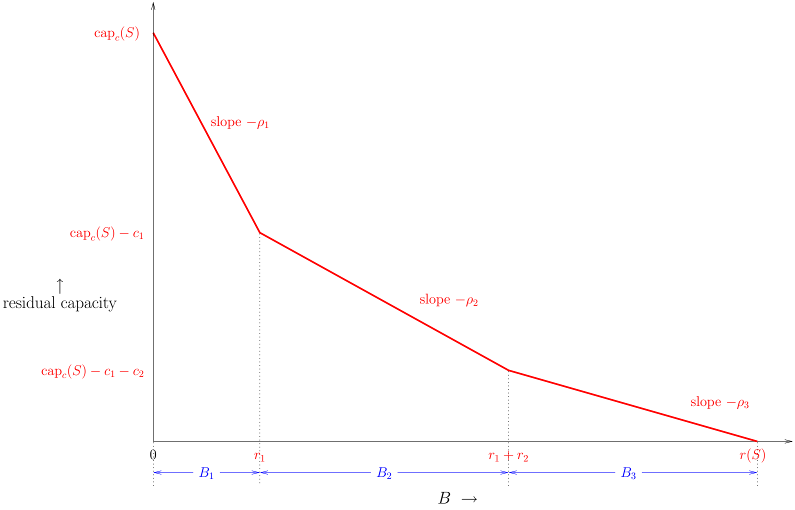


The interdiction curve for a fixed cut S

This curve is piecewise linear convex.



The overall interdiction curve: the B-profile

We overlay the cut-wise interdiction curves to get the overall curve.

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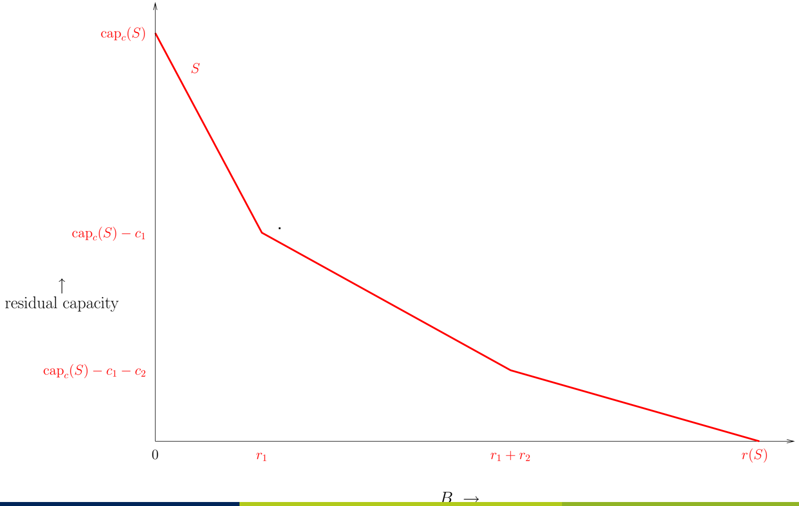

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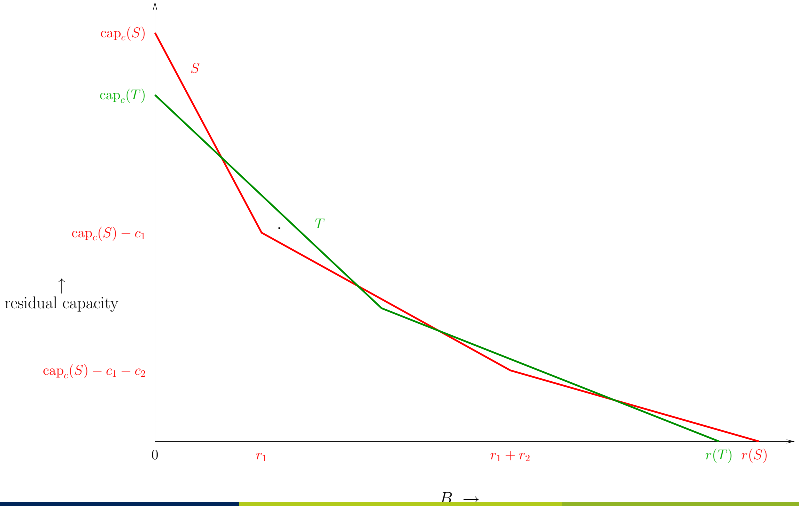

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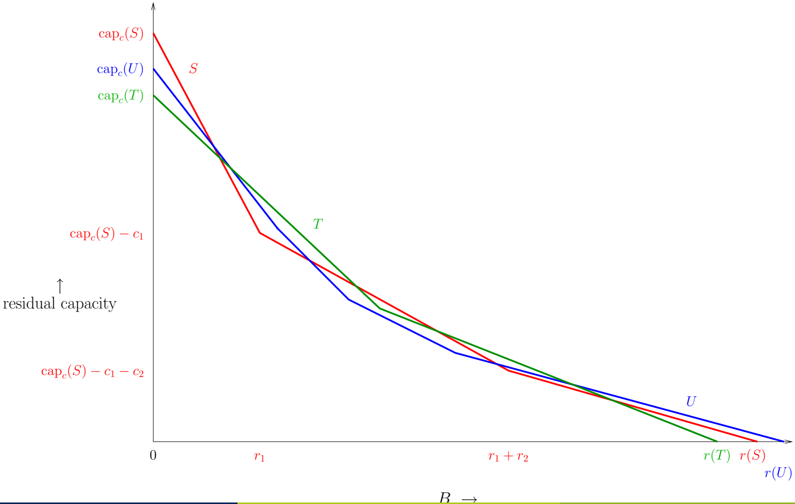

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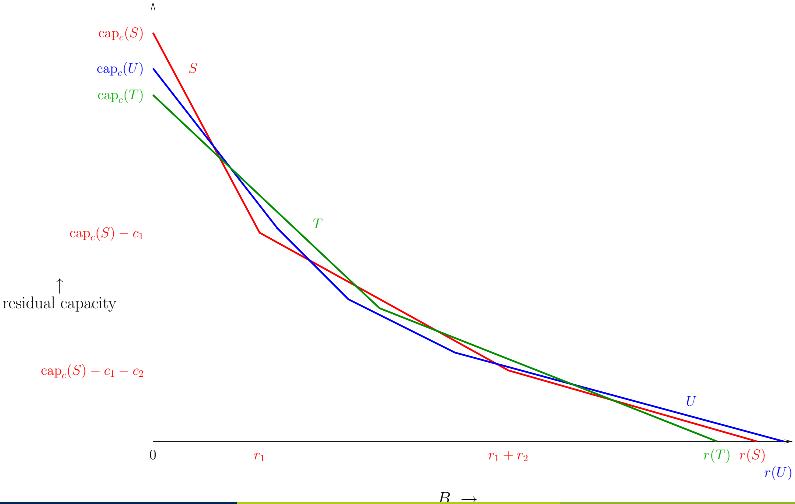
For a given value of B, we just select which S gives the minimum value at B, so the overall curve is the minimum of all the cut-wise curves.



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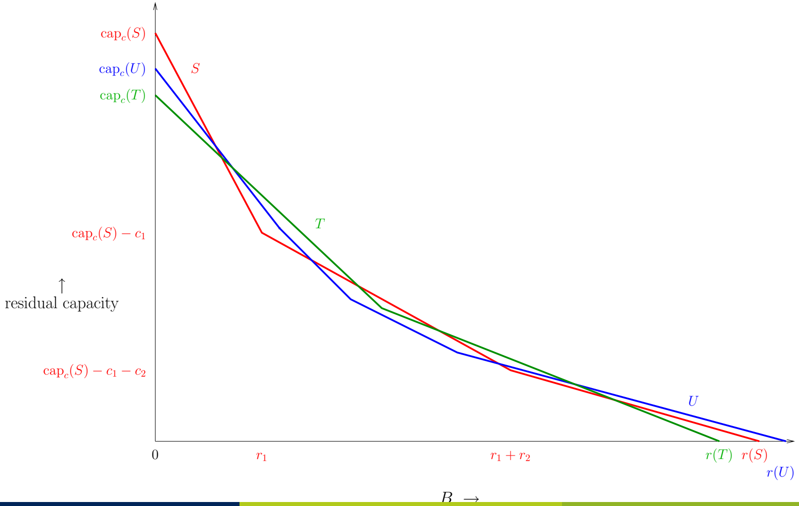
Unfortunately, the minimum of a bunch of convex curves is not in general convex.



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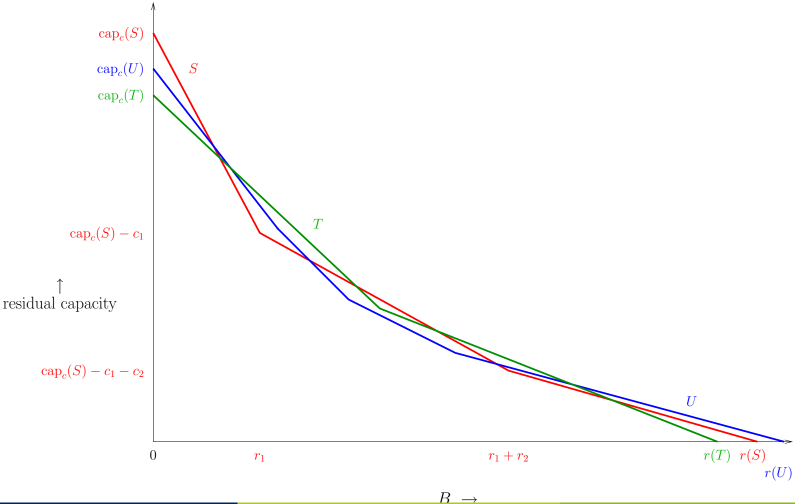
This is why Network Interdiction is NP Hard (Phillips '93; Wood '93).



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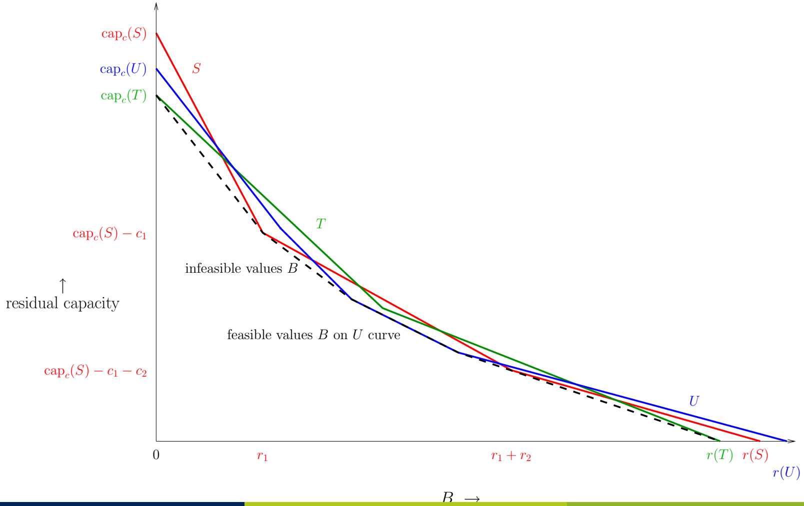
If we take the lower envelope, or convex hull, of the overall interdiction curve, we get something tractable, the B-profile.



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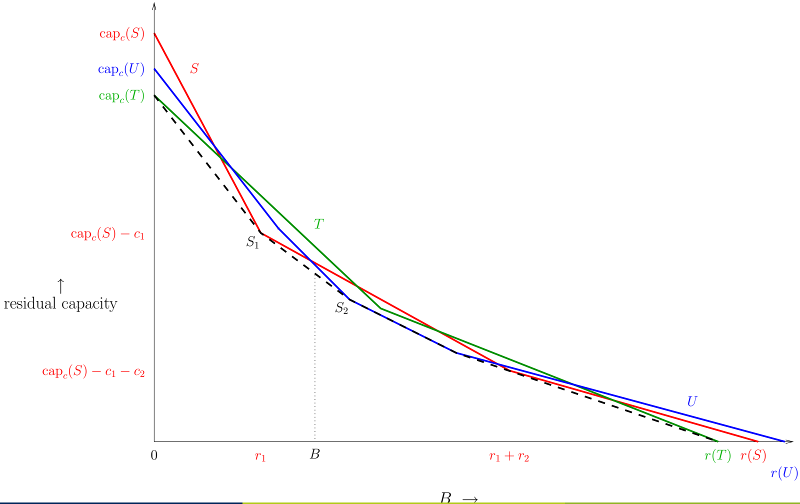
Now budget B corresponds to a convex combination of points coming from the interdiction curves of (one or) two cuts, $S_1 = S$ and $S_2 = U$.



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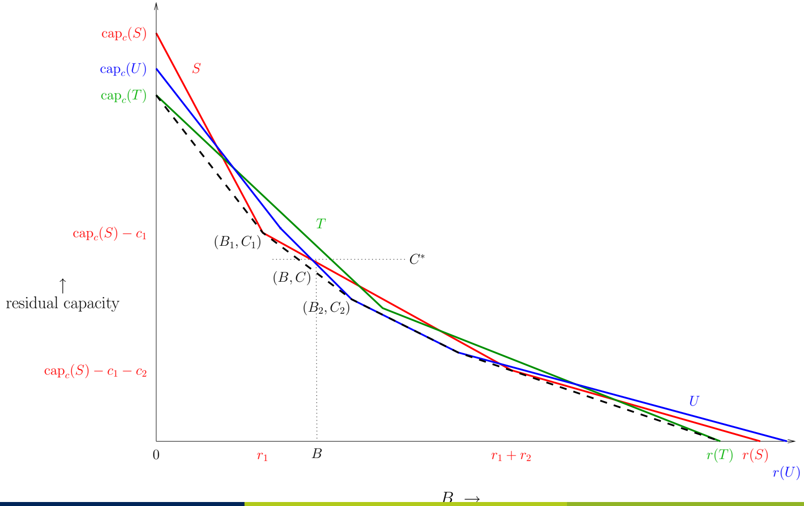
S_1 corresponds to breakpoint (B_1, C_1), S_2 to (B_2, C_2), and we have λ s.t. $B = \lambda_1 B_1 + \lambda_2 B_2$; define $C = \lambda_1 C_1 + \lambda_2 C_2 \leq C^* =$ opt. resid. capacity.



Linearizing the overall curve: the B-profile

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- The algorithmic question is then: Given B, how do we find S_1 and S_2? This shows that we also want B_1, B_2, C_1 and C_2.
- Burch et al write a linear program that can do it, but here we want a combinatorial algorithm to do it.

Outline

The linear program and its dual

- The normal min cut dual LP is

$$\begin{array}{rcll} \min \sum_{u \to v} c_{uv} y_{uv} & & & \\ \text{s.t. } d_u - d_v + y_{uv} & \geq & 0 & \text{for } u \to v \neq t \to s, \\ d_t - d_s + y_{ts} & \geq & 1 & \\ y_{uv} & \geq & 0 & \text{all } u \to v. \end{array}$$

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- Prize-collecting with a budget in place of a penalty.

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- Repeat the new LP with dual variables:

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x_{ts:}: & d_t - d_s + y_{ts} & \geq & 1 & \\
\lambda: & \sum_{u\to v} r_{uv} z_{uv} & \leq & B & \\
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x_{uv}: & \text{s.t. } d_u - d_v + y_{uv} + z_{uv} & \geq & 0 & \text{for } u \to v \neq t \to s, \\
x_{ts:}: & d_t - d_s + y_{ts} & \geq & 1 & \\
\lambda: & \sum_{u\to v} r_{uv} z_{uv} & \leq & B & \\
 & y_{uv},\ z_{uv} & \geq & 0 & \text{all } u \to v.
\end{array}
$$

- When we "primalize" this interdiction dual LP we get new primal variable λ corresponding to the dual constraint $\sum_{u\to v} r_{uv} z_{uv} \leq B$, and the z_{uv}'s give us a second set of capacities.

The linear program and its dual

- Repeat the new LP with dual variables:

$$
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$$
\begin{array}{rrcl}
 & \max_{x,\lambda}\ (x_{ts} - B\lambda) & & \\
d: & \text{s.t. conservation} & & \\
y_{uv}: & 0 \leq x_{uv} & \leq & c_{uv} \\
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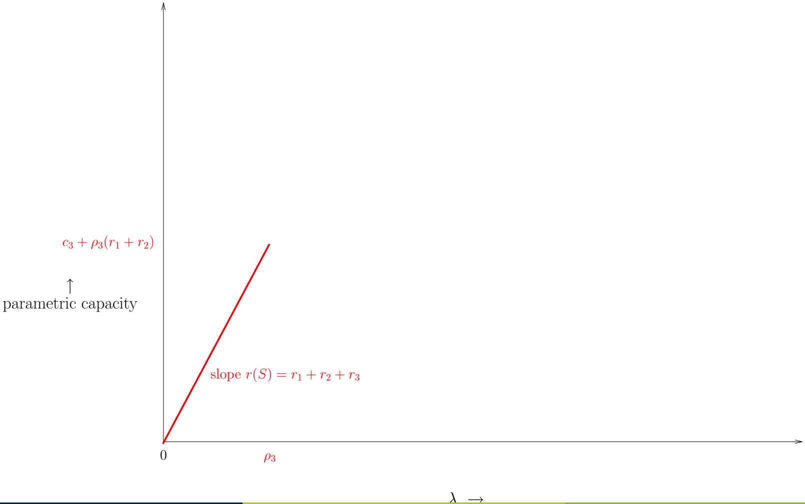
  a *parametric capacity* in the scalar parameter λ.

- So let's investigate the behavior of this parametric min cut problem.

Outline

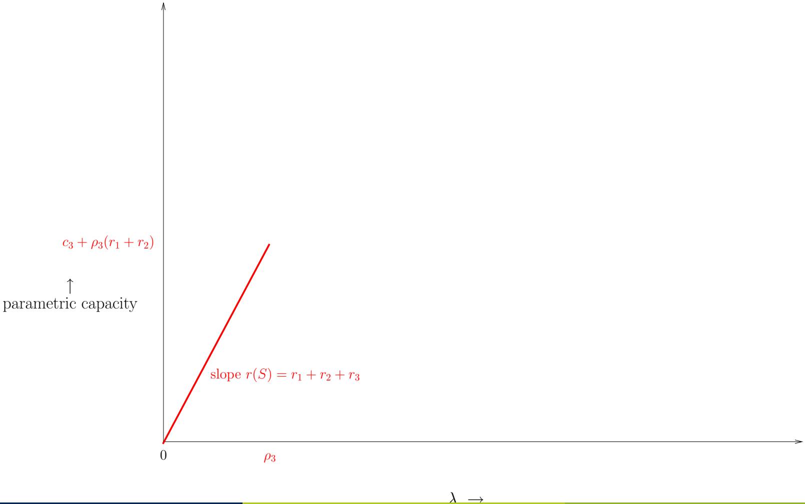
Parametric capacity of fixed cut S

When λ is small, $\text{cap}(S, \lambda) = \lambda \text{cap}_r(S)$.



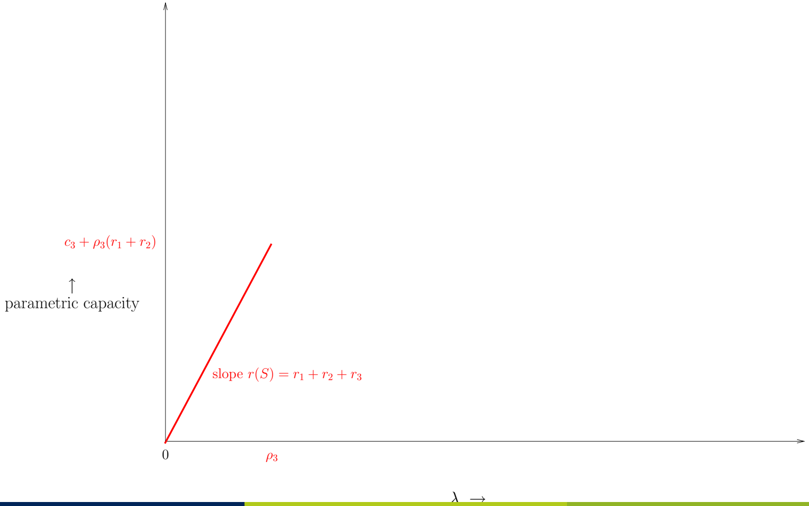
Parametric capacity of fixed cut S

This continues as long as $\lambda r_{uv} \le c_{uv}$ for all $u \to v \in \delta^+(S)$, or $\lambda \le \rho_{uv}$.



Parametric capacity of fixed cut S

Thus the first breakpoint is when λ hits $\min_{\delta^+(S)} \rho_{uv}$.



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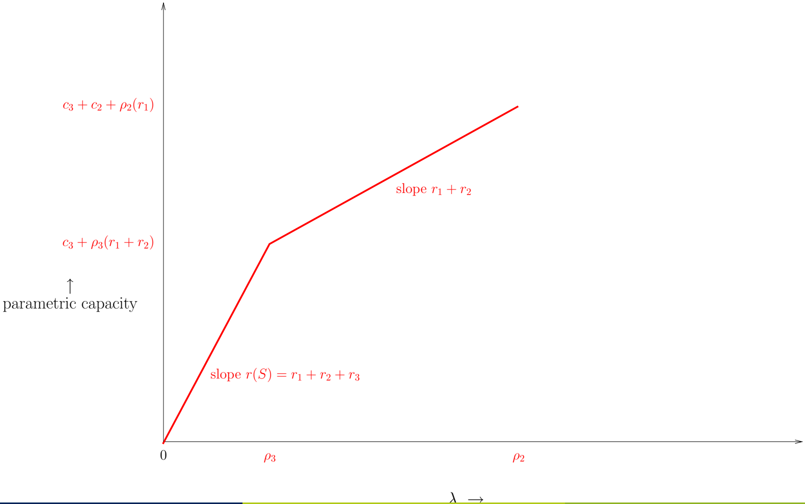

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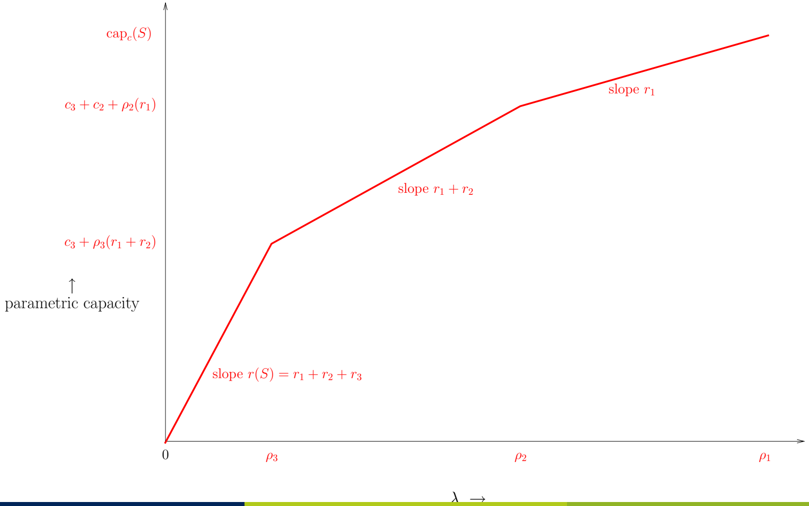

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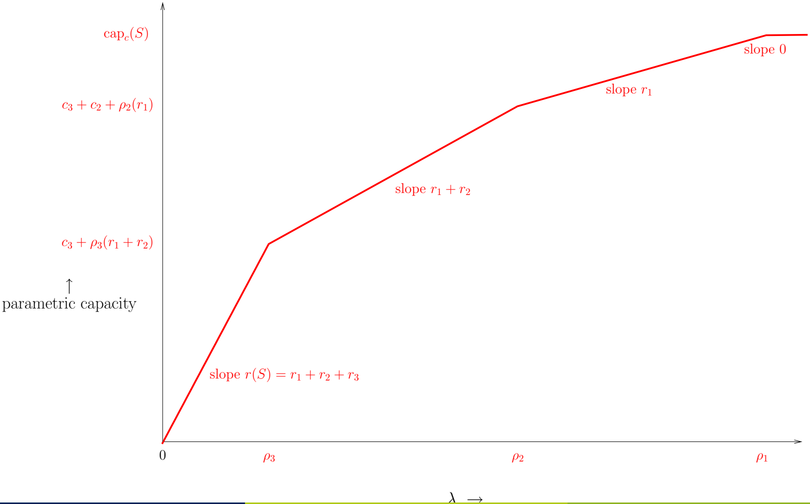
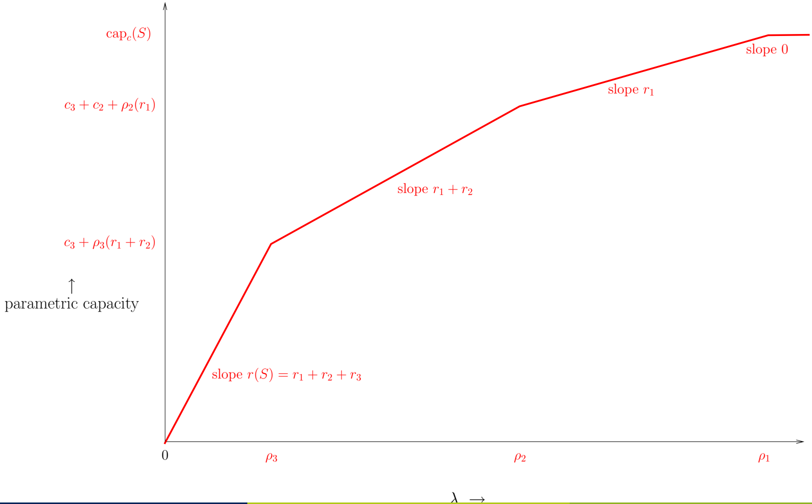


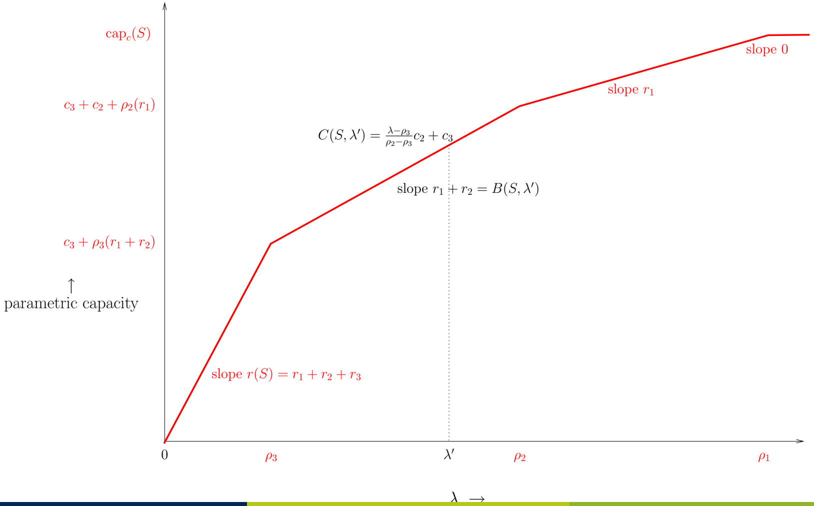
Parametric capacity of fixed cut S

The parametric capacity curve for S is piecewise linear concave.



Parametric capacity of fixed cut S

For a value λ' of λ we also get the local budget $B(S,\lambda')$ and local residual capacity $C(S,\lambda')$.



Conjugate duality between interdiction and parametric capacity for S

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- Thus breakpoints and slopes are interchanged between S's interdiction curve and its parametric capacity curve, though in reverse order and modulo a minus sign.
- In the language of conjugate duality, this is equivalent to saying that the parametric capacity curve $\text{cap}(S, \lambda)$ is the negative of the conjugate dual of the interdiction curve for S, evaluated at $-\lambda$.

The overall parametric capacity curve: the λ-profile

Now overlay the parametric capacity curves for all S.

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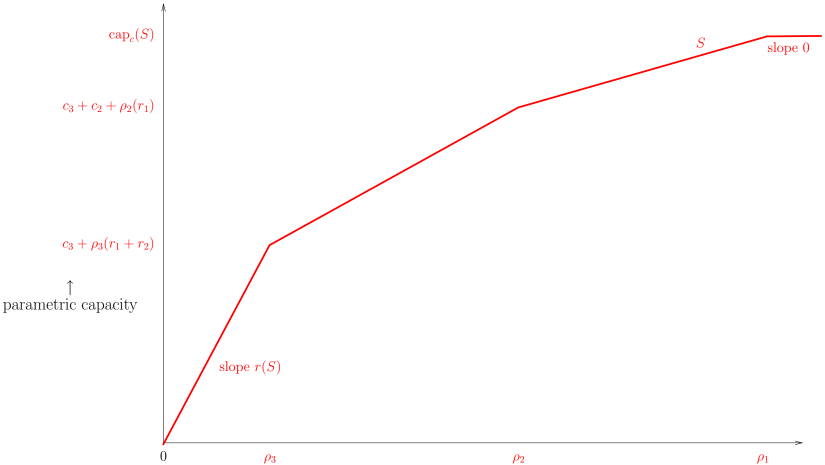

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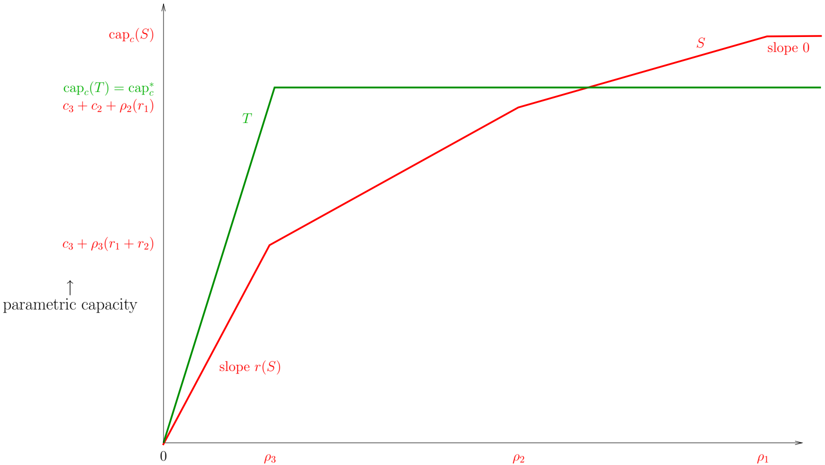

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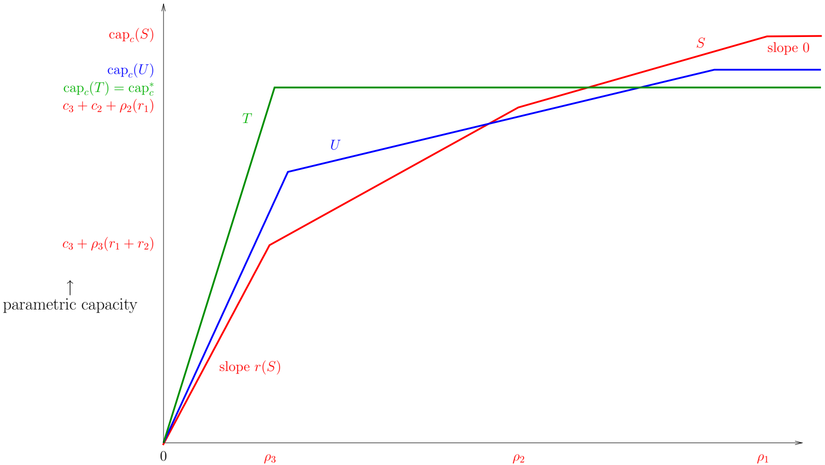
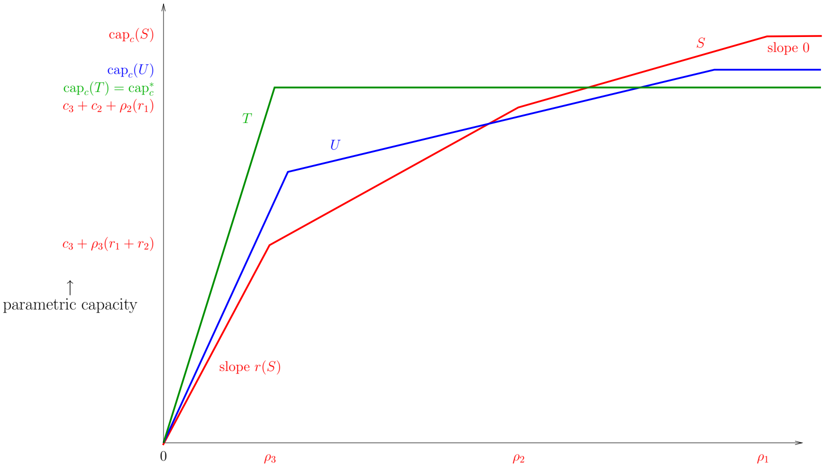


The overall parametric capacity curve: the λ-profile

For a fixed value of λ, we want to find the S whose parametric capacity at λ is minimum, so we just want the pointwise minimum of all these curves.



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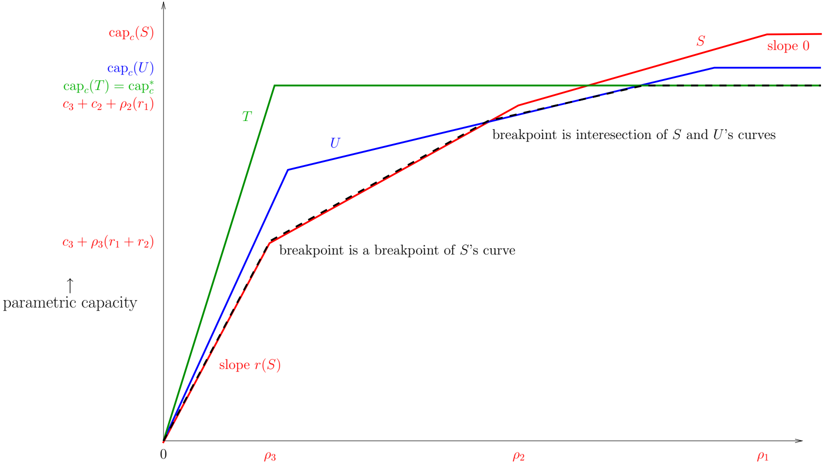
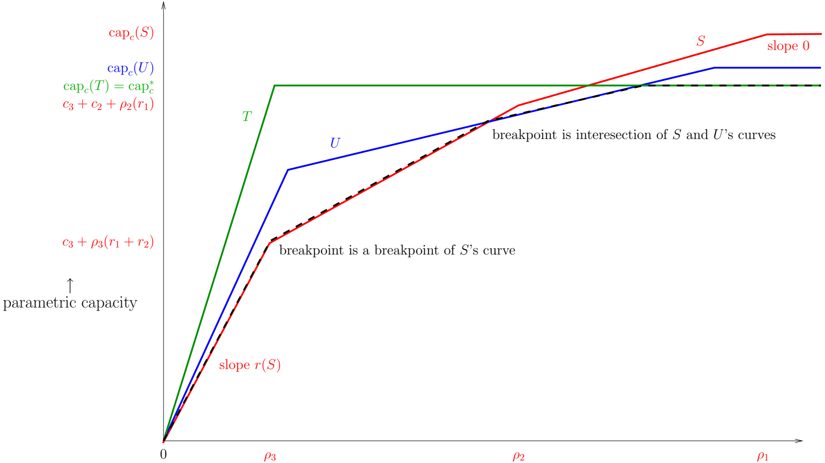


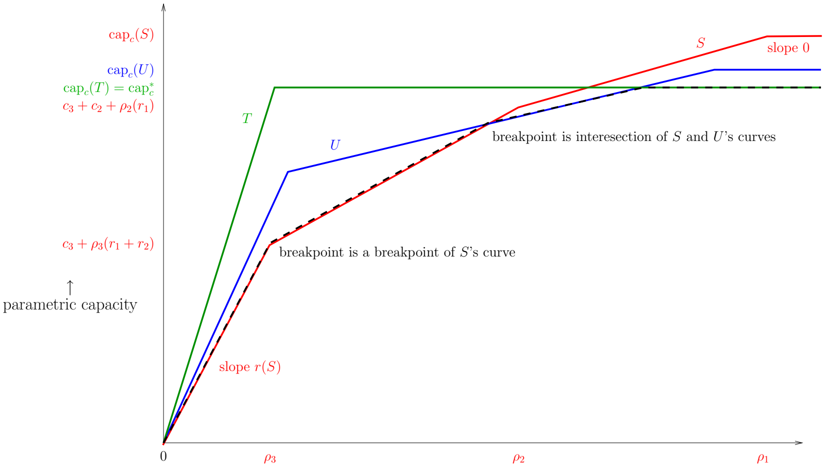
The overall parametric capacity curve: the λ-profile

Since the minimum of a bunch of concave curves is again concave, this time we do not need to linearize. We call this overall parametric capacity curve the λ-profile.



The overall parametric capacity curve: the λ-profile

We can compute things like $\text{cap}^*(\lambda)$ easily using parametric min cut technology.



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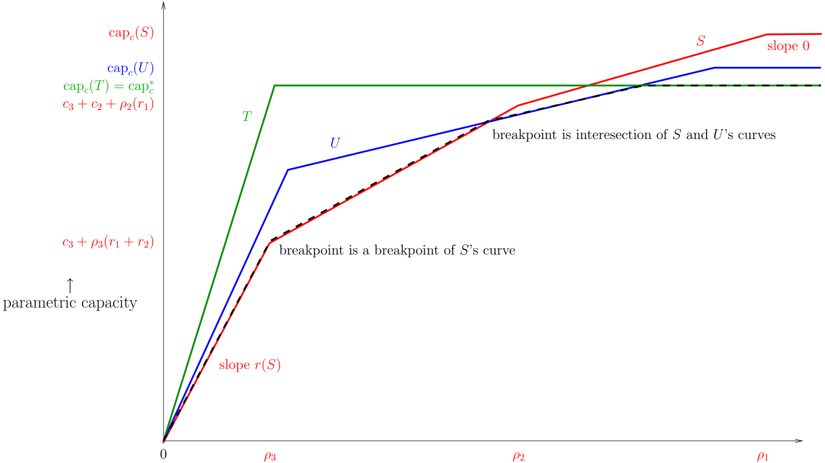
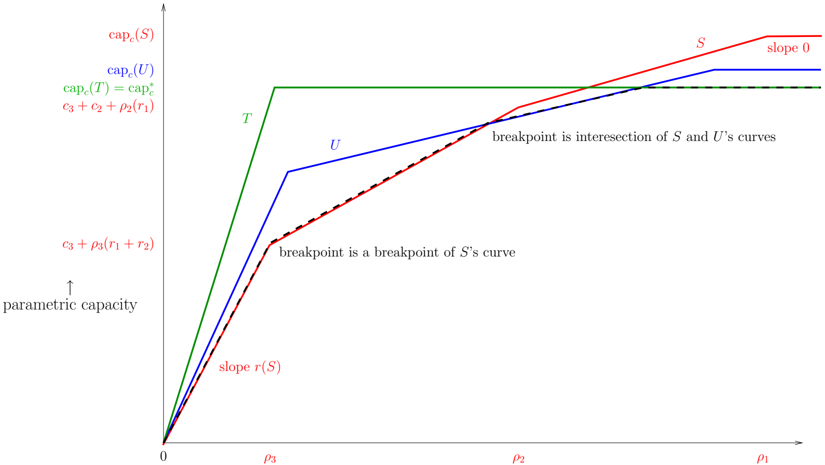
We can show that the conjugate duality between S's interdiction and parametric capacity curves carries over to conjugate duality between the B-profile and the λ-profile.



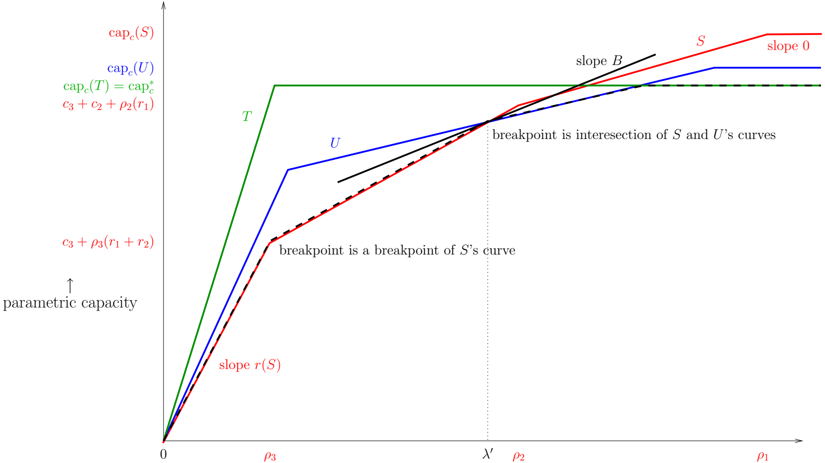
The overall parametric capacity curve: the λ-profile

Recall that to get our pseudo-approximation for a given B, we want to compute the two cuts S_1 and S_2 bracketing B on the B-profile.



The overall parametric capacity curve: the λ-profile

Conjugate duality implies that this is equivalent to finding a breakpoint λ' on the λ-profile whose adjacent slopes bracket B, here S and U; we also get $B_1 = B(S_1, \lambda')$, $C_1 = C(S_1, \lambda')$, $B_2 = B(S_2, \lambda')$, and $C_2 = C(S_2, \lambda')$.



Outline

The key breakpoint subproblem

- Notice that any breakpoint $\hat{\lambda}$ of the λ-profile is defined by the intersection of a segment to its left coming from cut $S^-(\hat{\lambda})$ with local slope $\mathrm{sl}^-(\hat{\lambda})$, and a segment to its right coming from cut $S^+(\hat{\lambda})$ with local slope $\mathrm{sl}^+(\hat{\lambda})$, with $\mathrm{sl}^-(\hat{\lambda}) > \mathrm{sl}^+(\hat{\lambda})$ by concavity.

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- So let's just concentrate on finding λ_B.

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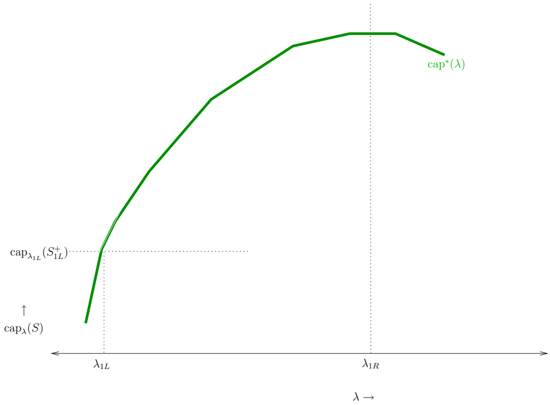
- Can we do better?

Discrete Newton gives a better algorithm

Set $\lambda_L = 0$ and $\lambda_R = \mathrm{cap}_r^*$ as before. Denote $\mathrm{sl}^+(\lambda_L)$ by sl_L^+ and $\mathrm{sl}^-(\lambda_R)$ by sl_R^-.



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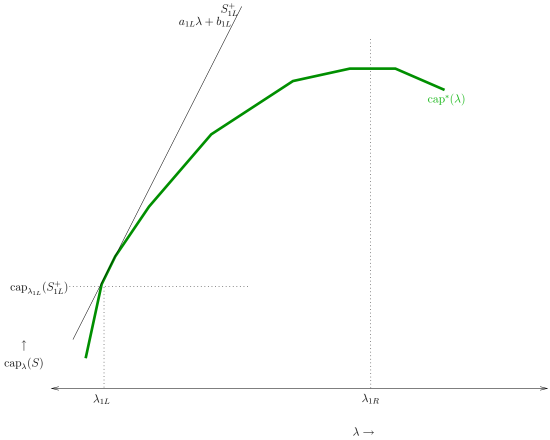

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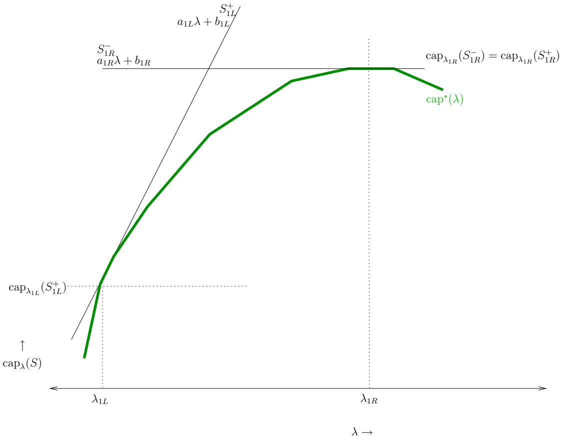

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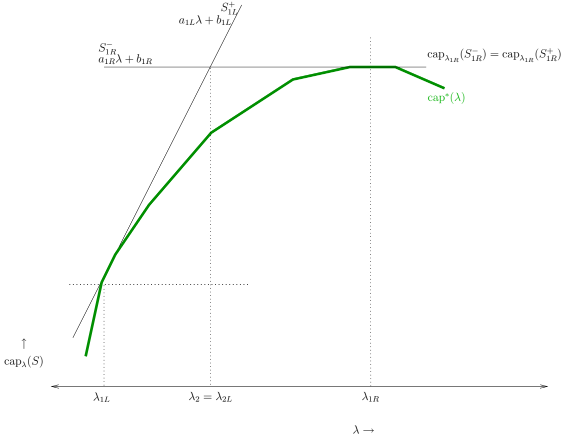
Compute $\hat{\lambda}$ as the intersection of the line of slope sl_L^+ through $(\lambda_L, \mathrm{cap}^*(\lambda_L))$, and the line of slope sl_R^- through $(\lambda_R, \mathrm{cap}^*(\lambda_R))$, a max flow w.r.t. $\hat{\lambda}$, and $\mathrm{sl}^-(\hat{\lambda})$ and $\mathrm{sl}^+(\hat{\lambda})$.



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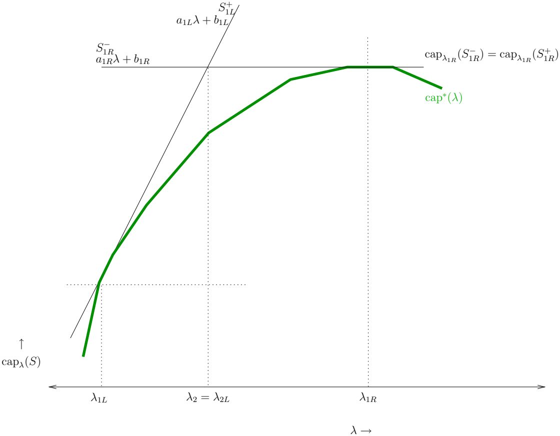

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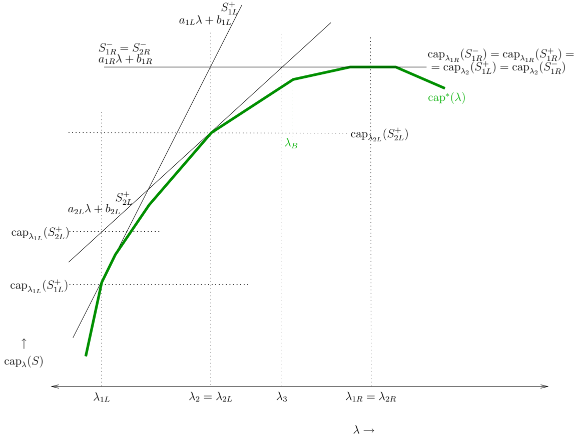

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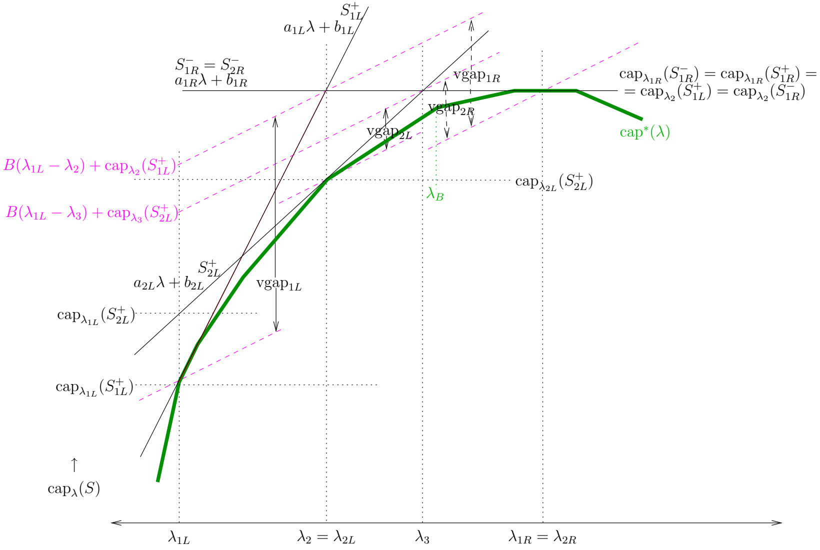
- Also define slgap_L to be $\text{sl}_L^+ - B$ and slgap_R to be $B - \text{sl}_R^-$.

vgap illustrated



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  - The better weakly polynomial bound is $O\left(\frac{\log(nD)}{1+\log\log(nD)-\log\log n}\right)$.
  - Sometimes there is an $O(m)$ bound on the number of iterations.

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- Indeed, this Newton-B algorithm and its analysis works for any concave (or convex) function, even continuous ones.

Outline

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- As before we could solve this via LP, but we'd prefer a combinatorial algorithm.
- Interdiction already gets complicated with two parameters, so let's consider a simpler multiple parameter scheduling problem instead.

Chen's '94 scheduling problem

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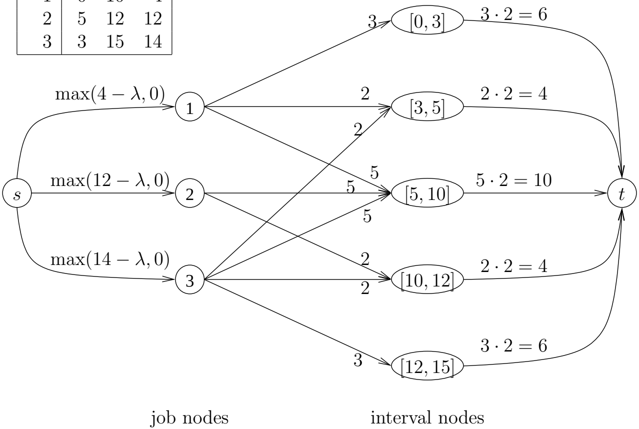
- Now we want to minimize λ such that there exists a flow saturating all residual job arcs.

Chen's scheduling problem: example

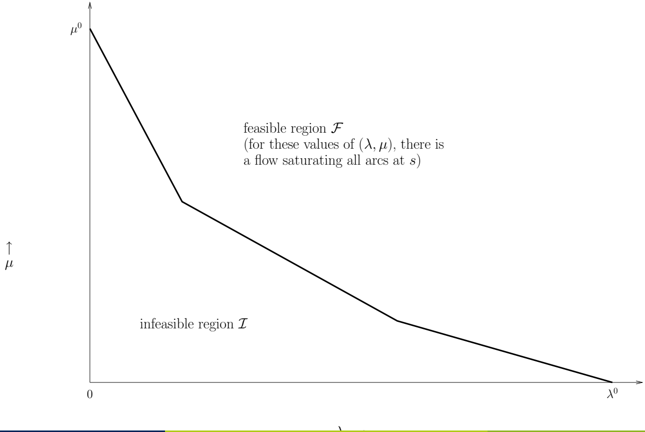
This single-parameter version can be solved using Gallo-Grigoriadis-Tarjan (GGT) '89 parametric min cut in $O(1)$ max flow time.

job	r_j	d_j	p_j
1	0	10	4
2	5	12	12
3	3	15	14



Two-parameter Chen

Suppose now that there are two ways to outsource, λ and μ such that if we pay $\$\lambda + \$\mu$, we reduce p_j to $\max(0, p_j - a_j\lambda - b_j\mu)$.
In the (λ, μ) plane there is a piecewise linear convex curve separating feasible points from infeasible ones.



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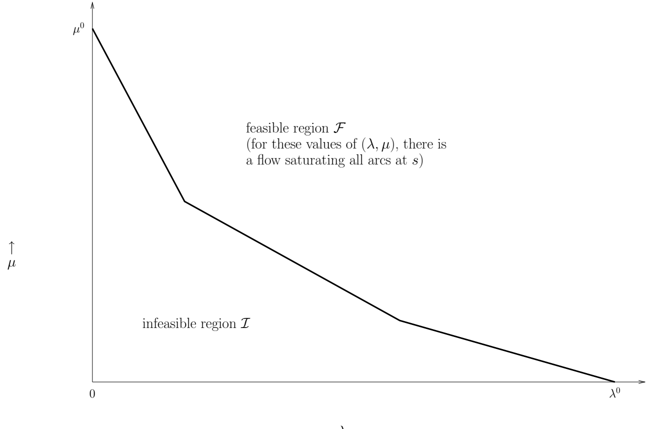

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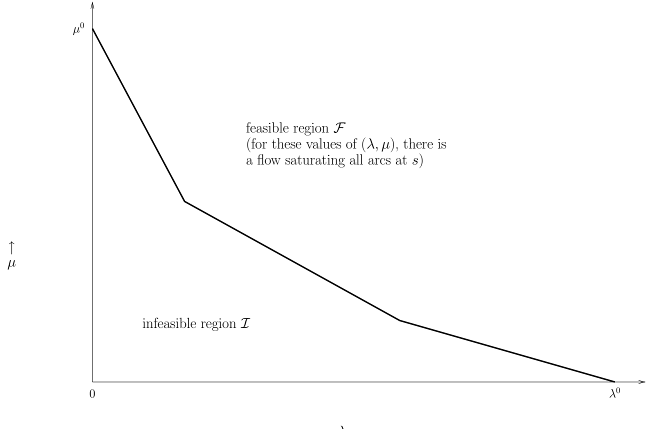

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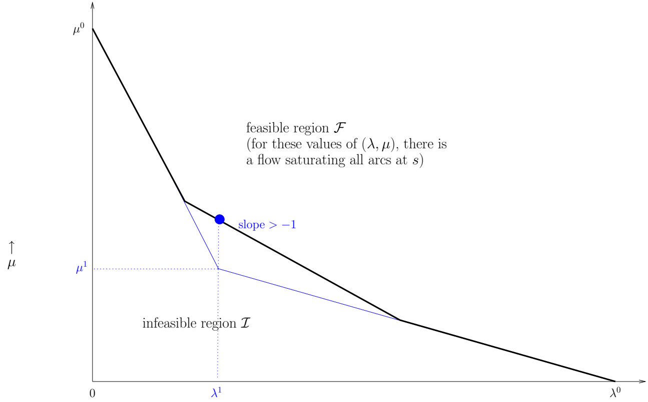

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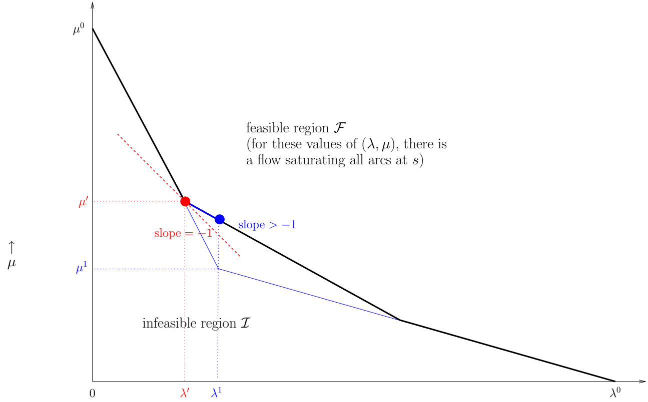

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- This generalizes to any fixed number of parameters.
- Open Question: LP is polynomial even when the number of parameters is not fixed. Can we get a combinatorial algorithm then?

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- **Corollary:** Min cuts are increasing along any non-decreasing curve (chain) in $\mathbb{R}^2_+$.
- Open Question: When capacities are (piecewise) linear, how many different min cuts can we have over all (λ, μ)?

Any questions?

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Comments?