# The Euclidean k-Supplier Problem

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Baruch Schieber (IBM)
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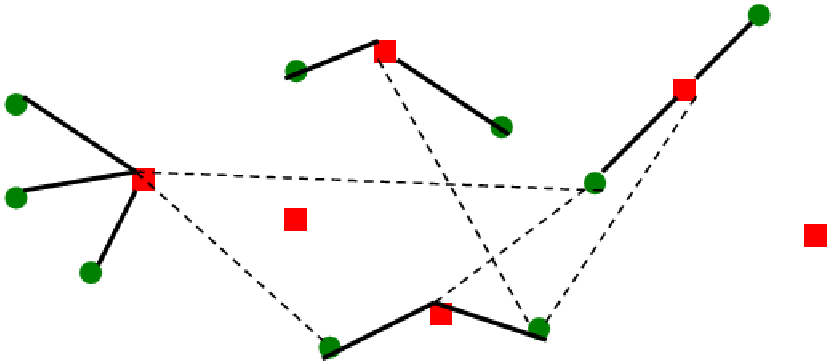
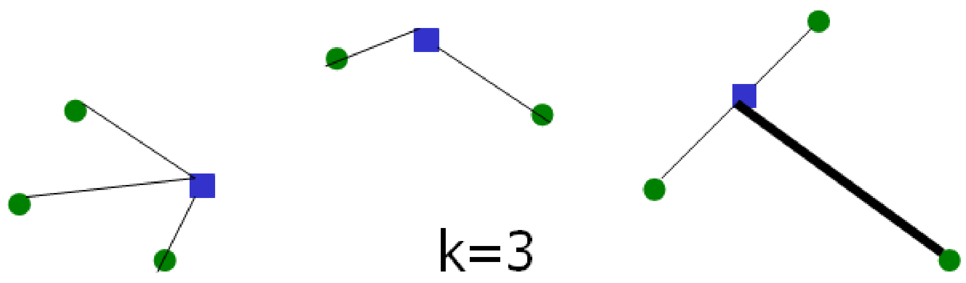
# Facility location problems

- Applications
  - plant location
  - placing servers in network
  - clustering…
- Clients and potential facilities
- Metric distances



# k-Center

- Metric (V,d), $d : V \times V \to R_+$
- Open k centers S
  - Each vertex connects to nearest center
- Minimize $\max_{u \in V} d(u,S)$



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- Metric (V,d), $d : V \times V \to R_+$
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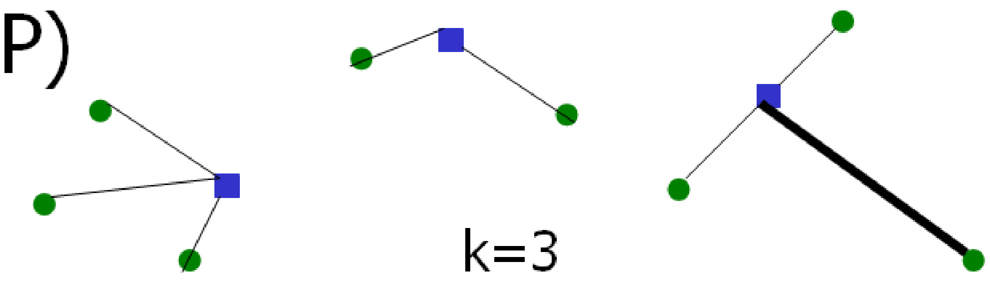
- Minimize $\max_{u \in V} d(u,S)$
- 2-approximation algorithms

  [Hochbaum, Shmoys '85] [Gonzalez '85]
  - Best possible (P≠NP)



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- Distinct sets of clients C, facilities F
  - metric ($C \cup F$, d)
- Open k facilities $S \subseteq F$ as centers
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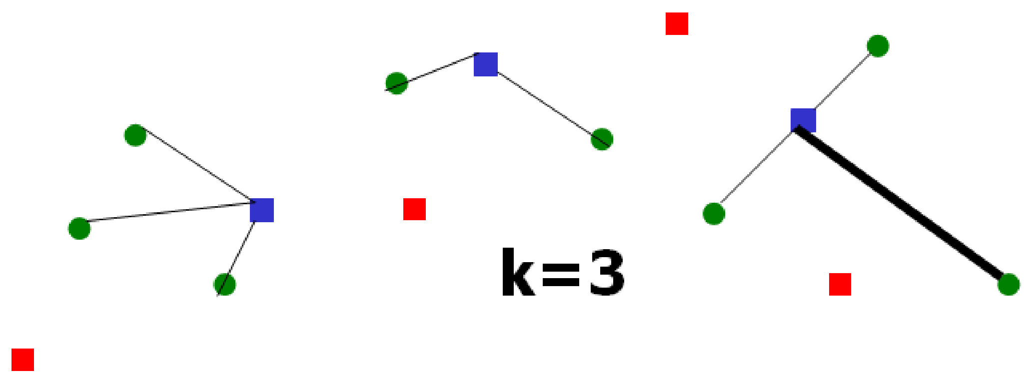

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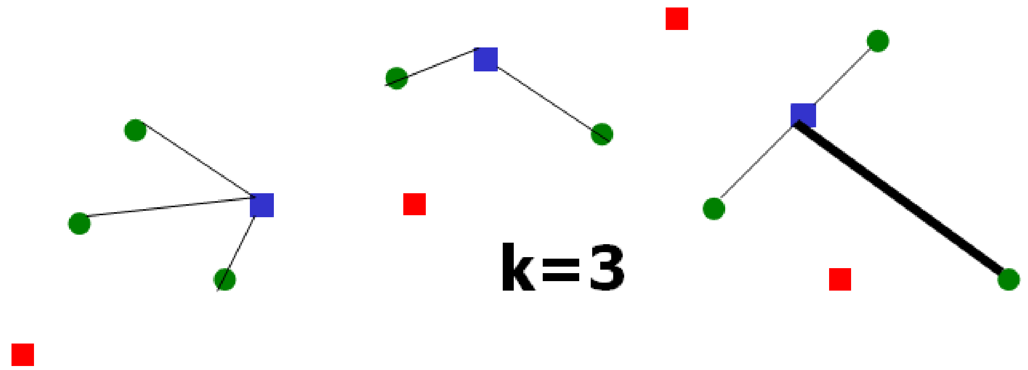
- 3-approximation algorithm [Hochbaum, Shmoys '86]
  - Best possible ($P \neq NP$)



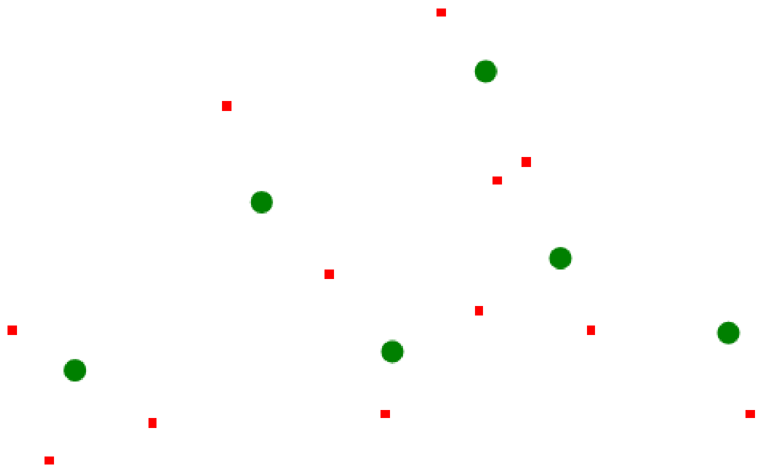
# Outline

- k-Supplier on general metrics
- Euclidean k-Supplier
- Fast approximation algorithm

# Algorithm for k-Supplier

[Hochbaum, Shmoys '86]

- Guess optimal value L



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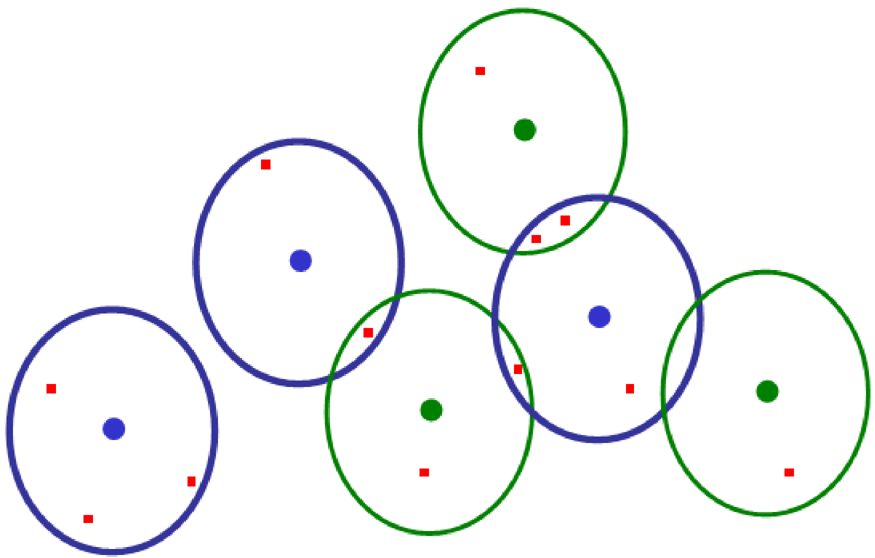

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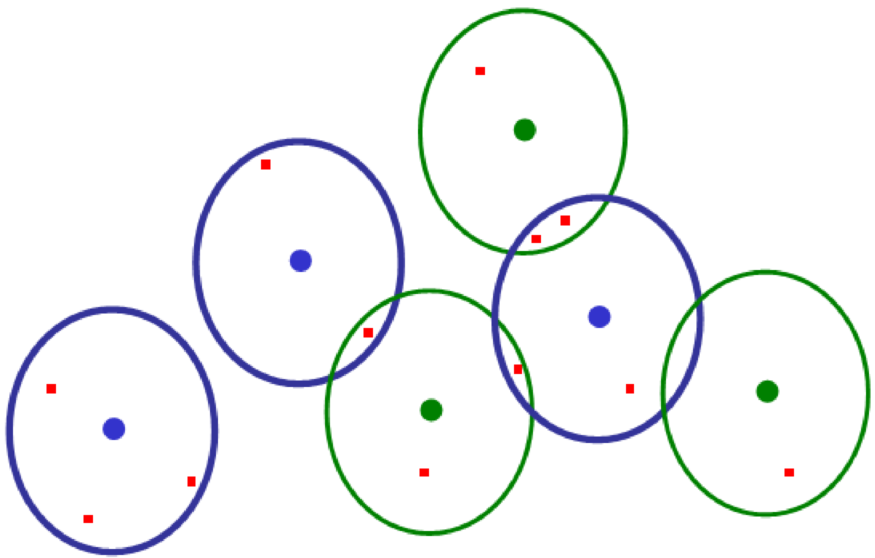

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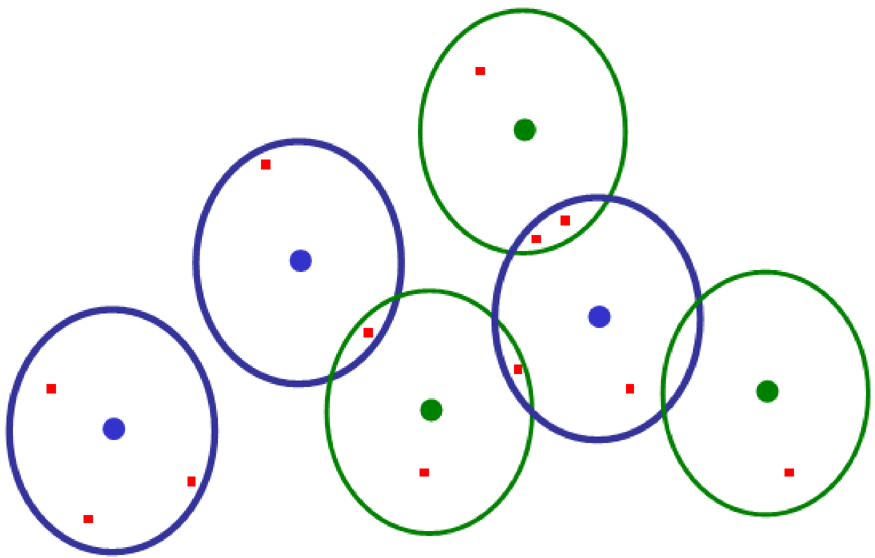

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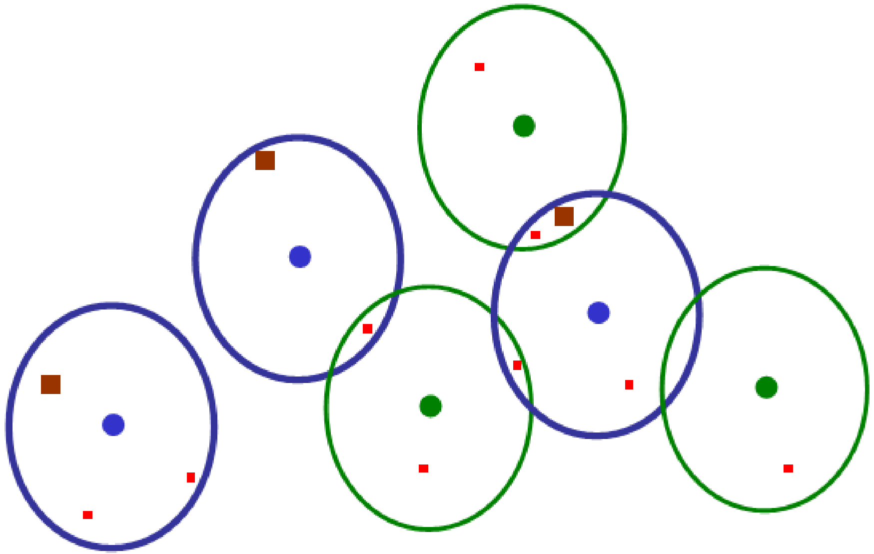

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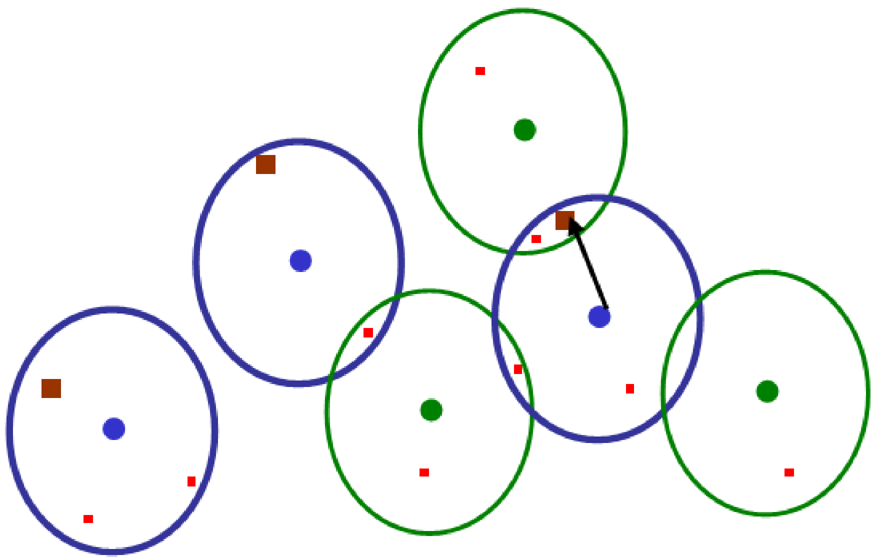

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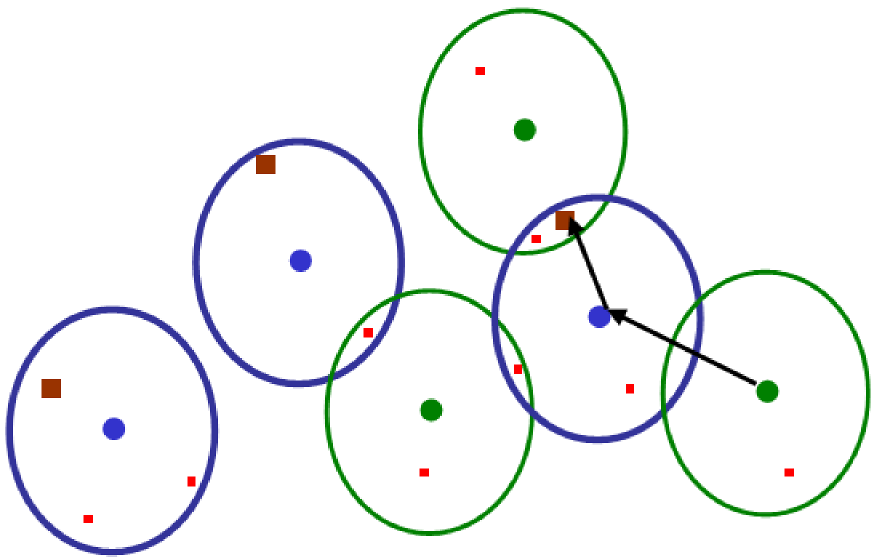

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# Hardness for k-Supplier

[Hochbaum, Shmoys '86]

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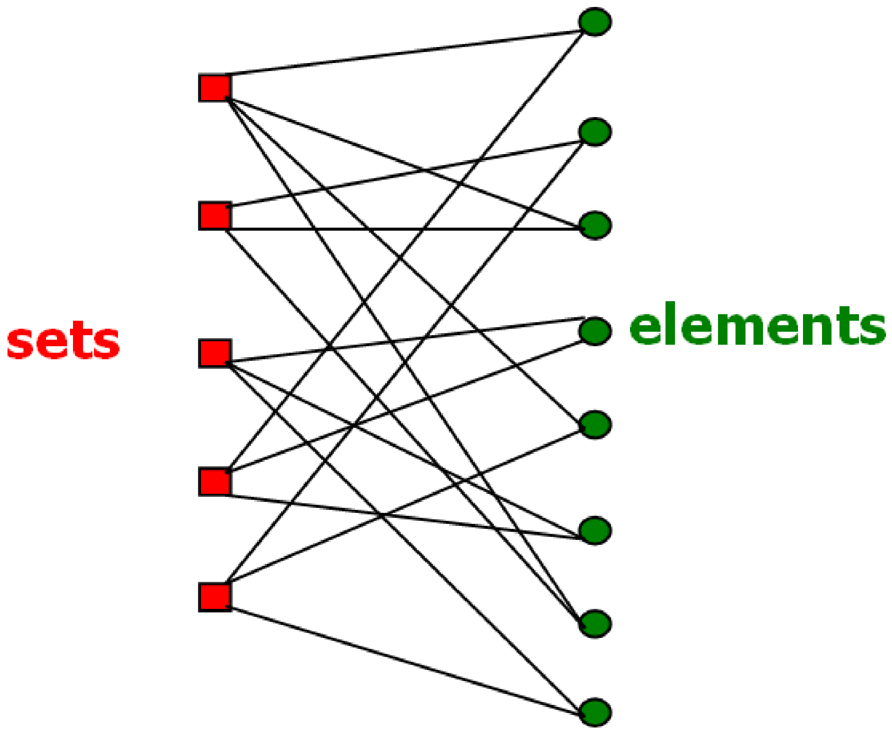

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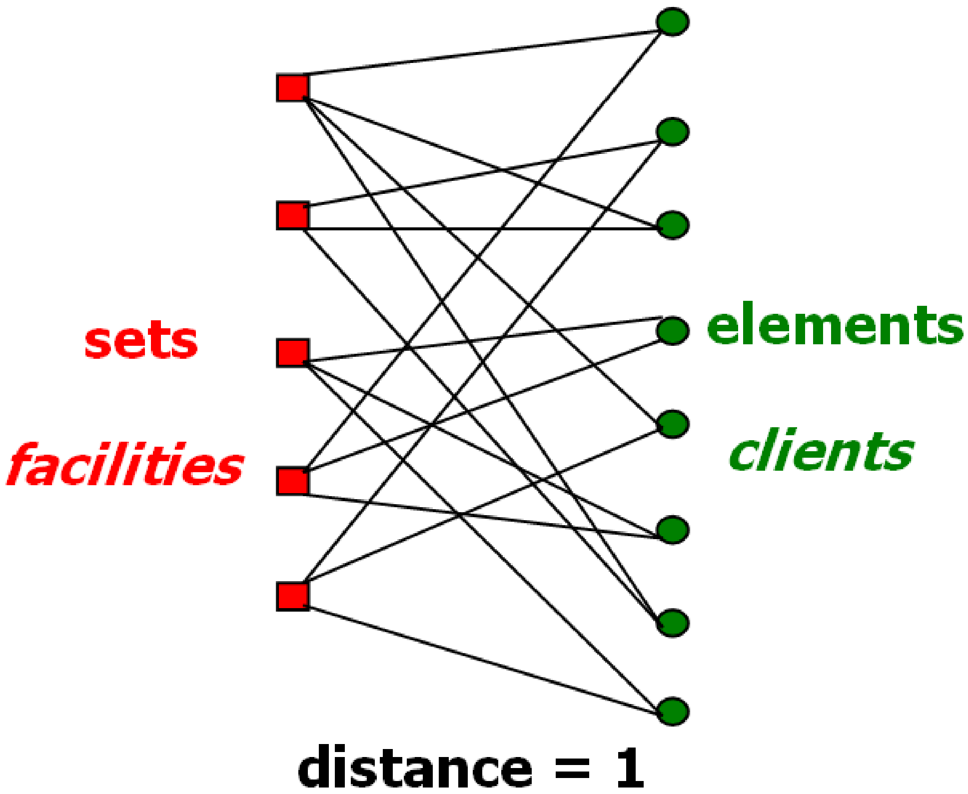

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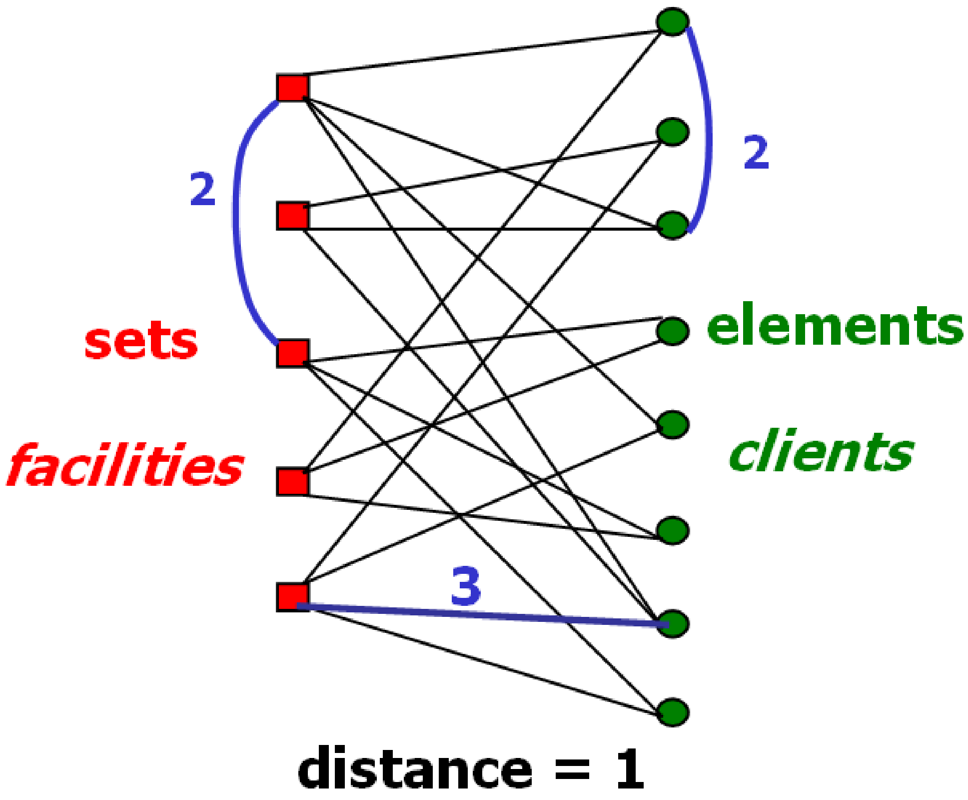

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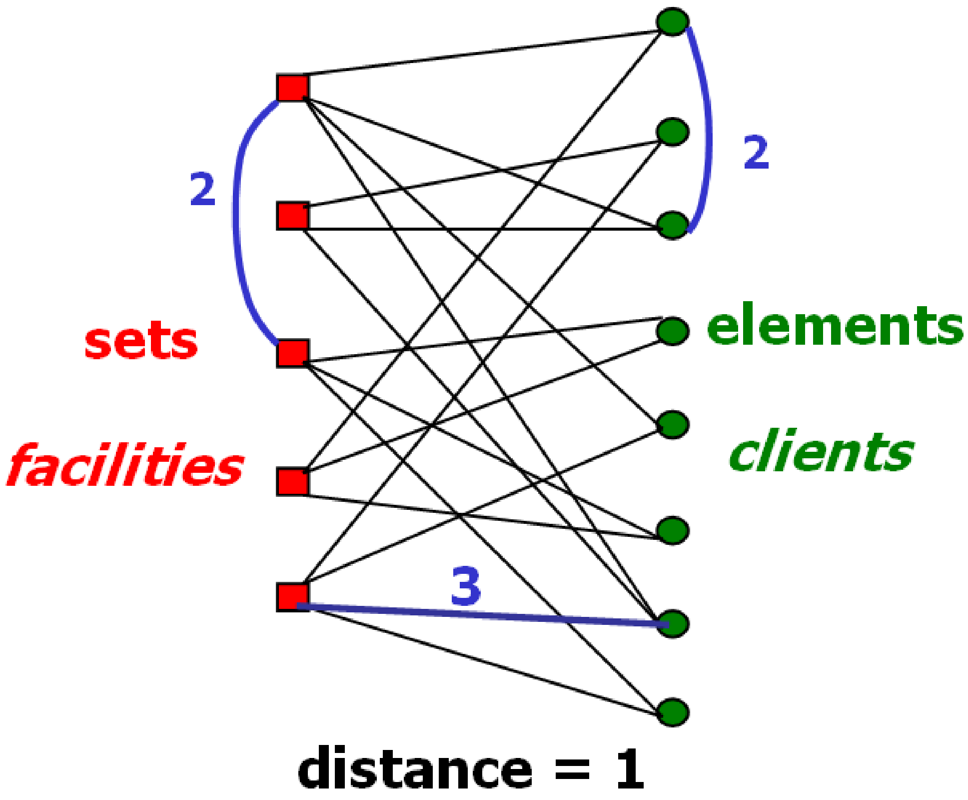


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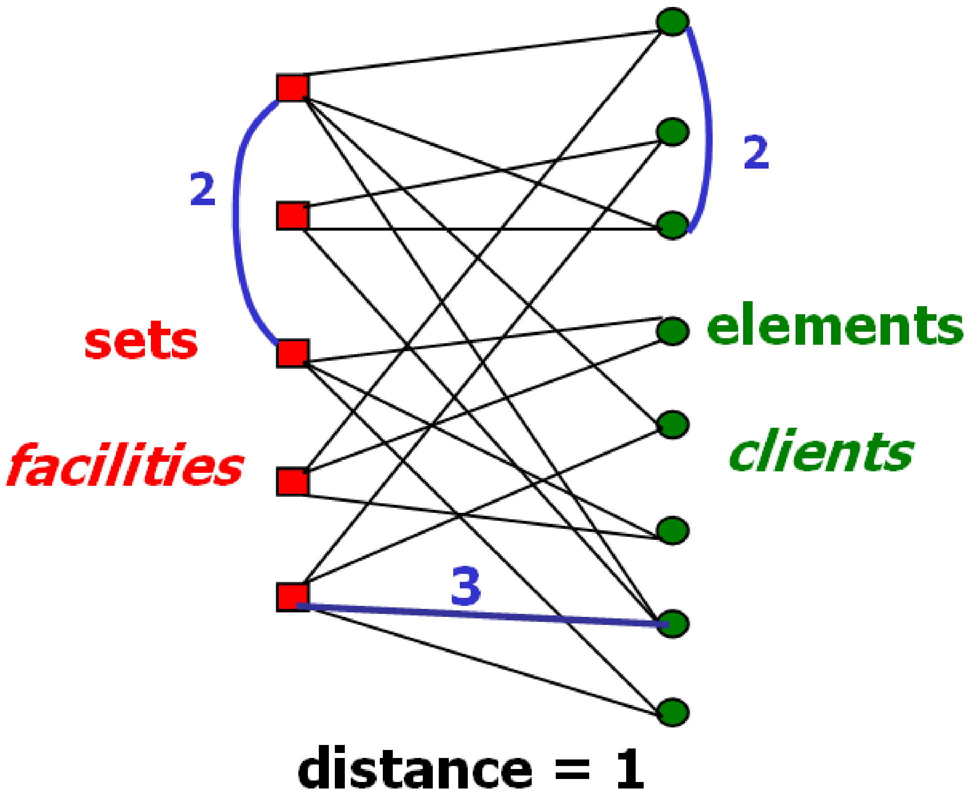



$\mathrm{OPT(SC)} \leq k \Rightarrow \mathrm{OPT}(\textit{k-Supp}) = 1$

$\mathrm{OPT(SC)} > k \Rightarrow \mathrm{OPT}(\textit{k-Supp}) = 3$

# Euclidean k-Supplier

- Points in $R^d$

- Euclidean distance
  - $d(u,v) = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \ldots + (u_d-v_d)^2}$

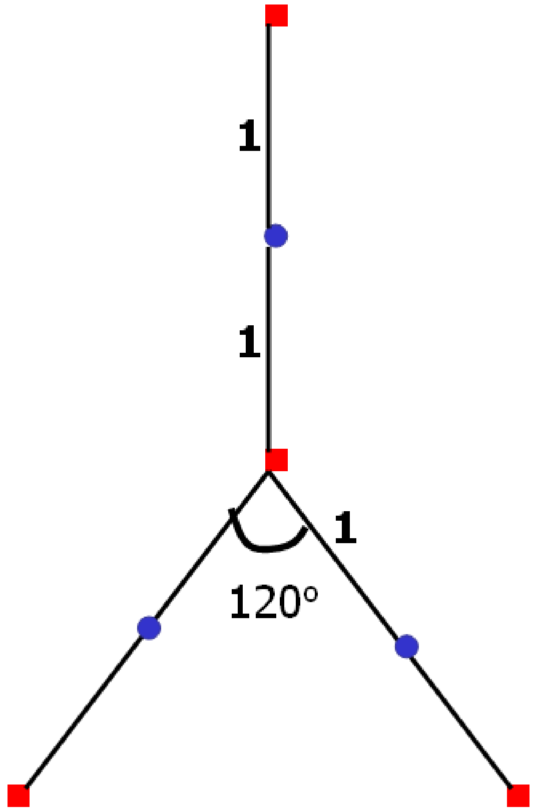
- Natural metric in many applications

*Can we do any better?*

# Euclidean k-Supplier Hardness

**Thm**: NP-hard to approximate better than $\sqrt{7}$

- From degree 3 planar vertex cover [Feder, Greene '88]



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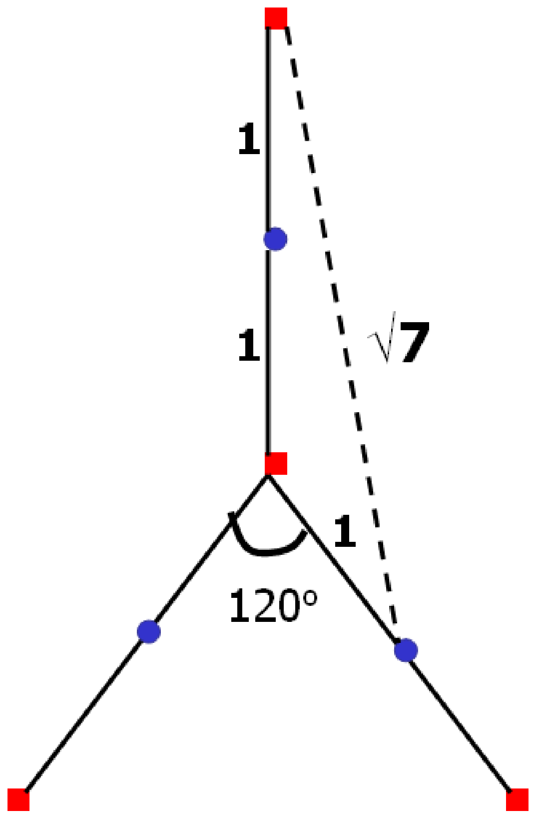

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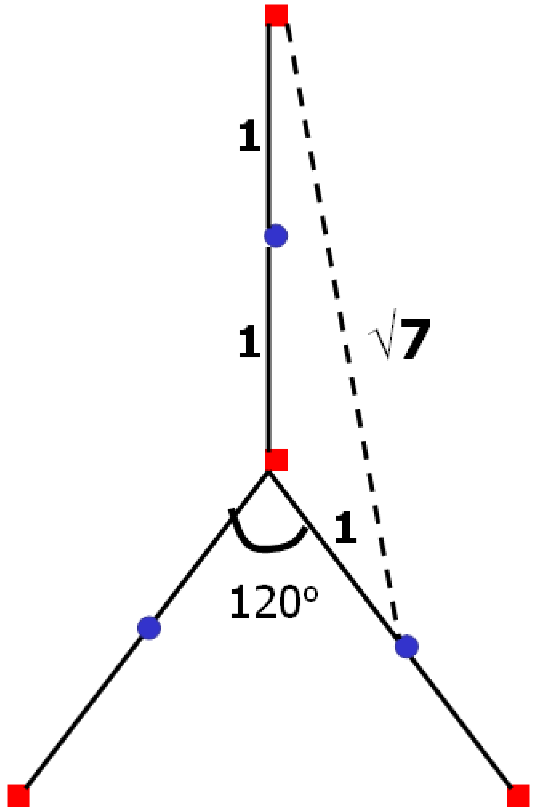


OPT(VC) $\leq$ k $\Rightarrow$ OPT(k-Supp) = 1

OPT(VC) **>** k $\Rightarrow$ OPT(k-Supp) $\geq$ √7

# Results

## Euclidean k-Supplier

- $\sqrt{3}+1 < 2.74$ approximation algorithm
  - Any number of dimensions
  - Running time ~$O(d \cdot n^2)$
  - $\sqrt{7} > 2.64$ hardness [Feder, Greene '88]

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Euclidean k-Supplier

- $\sqrt{3}+1 < 2.74$ approximation algorithm
  - Any number of dimensions
  - Running time $\sim O(d \cdot n^2)$
  - $\sqrt{7} > 2.64$ hardness [Feder, Greene '88]

- Fast 2.965 approximation algorithm
  - Time $O(n \cdot \log^2 n)$, $d=O(1)$ dimensions
  - Previously 3-approx in $O(n \cdot \log k)$

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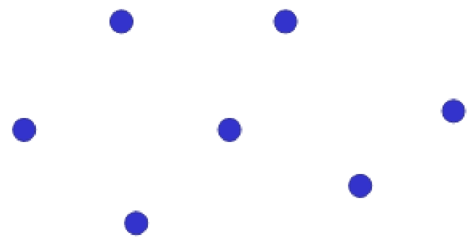
- *Maximal* clients **S** of pairwise distance > $\sqrt{3}$ L



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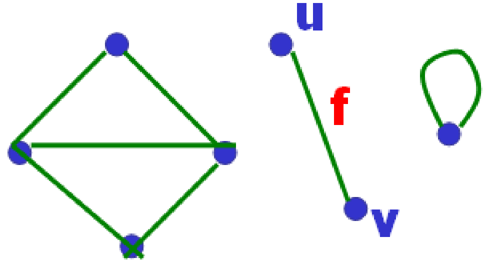
**graph G**



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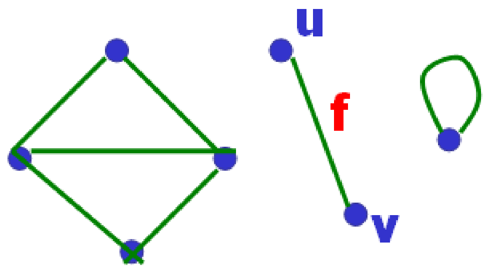


**Edge cover**: min number of edges to cover all vertices

Poly-time algorithm [Edmonds '65]...

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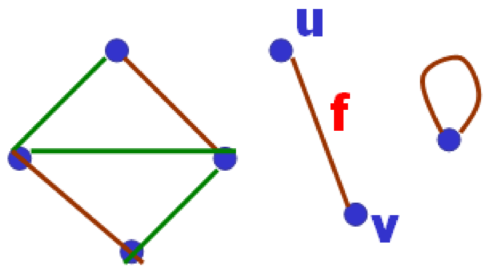


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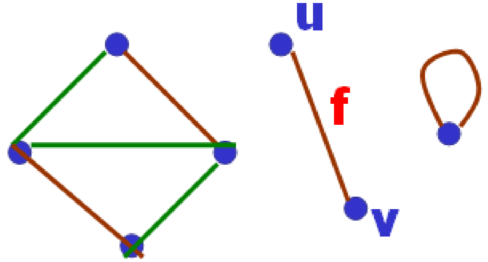


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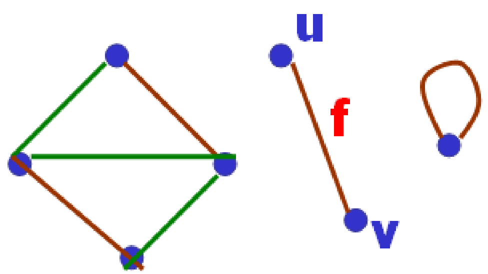
***To show***
  - EC* > k ⇒ OPT > L
  - EC* ≤ k ⇒ ALG ≤ (√3+1) · L

**graph G**

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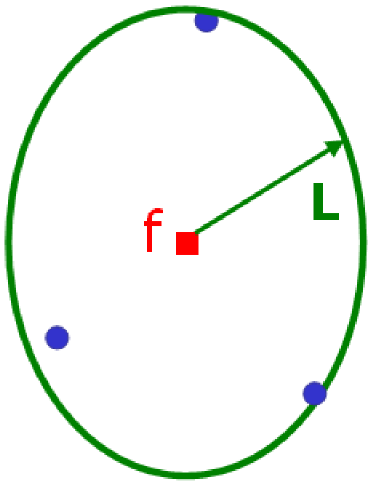

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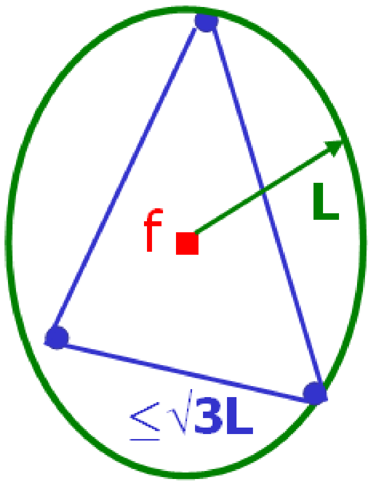

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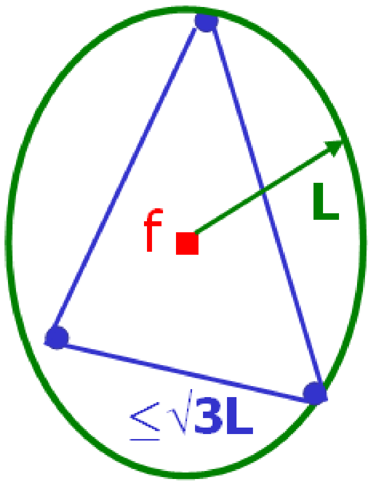


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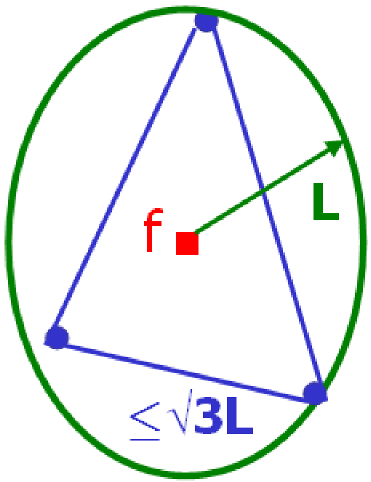
- k-Supplier[S] of value L ≡ edge cover on G

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- Construct graph G in $O(dn^2)$
  - naïve method
- Solve edge cover in $O(n^{1.5})$
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- Overall: $O(dn^2 \cdot \log n)$

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- Improved runtime for $d=O(1)$

# Faster implementation

- Approximate nearest neighbor
  - [Arya, Mount, Netanyahu, Silverman, Wu '98]
  - 1+$\epsilon$ approx. nearest neighbors
  - O(log n) time per query: NN, add, delete
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  - Use additional structure in **G** ?

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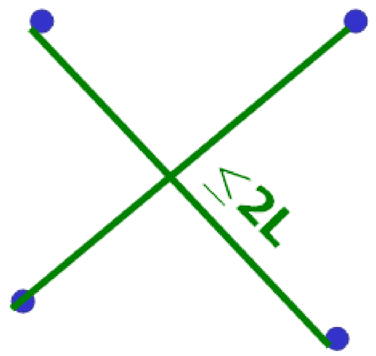

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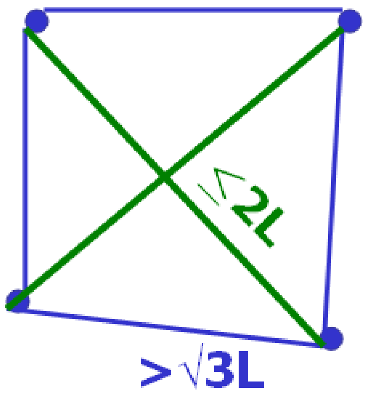

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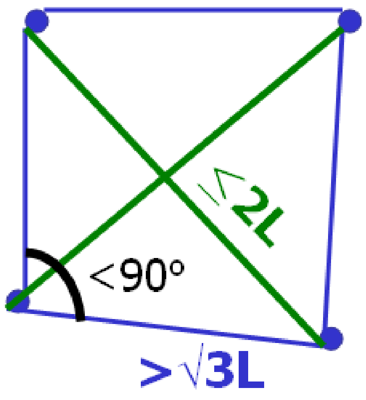

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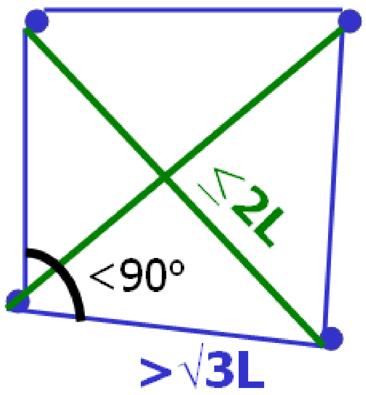


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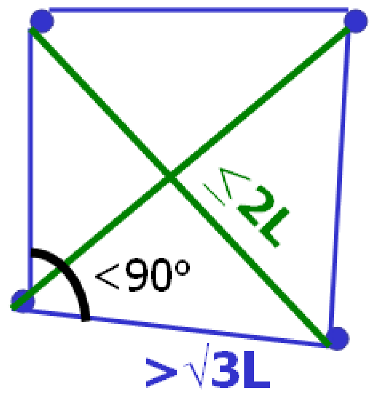


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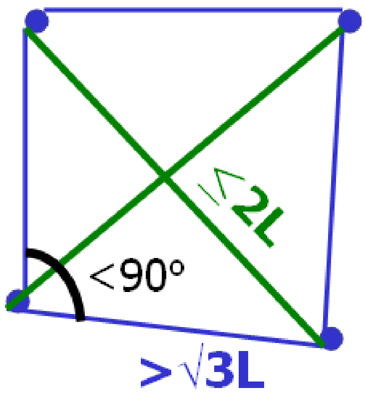


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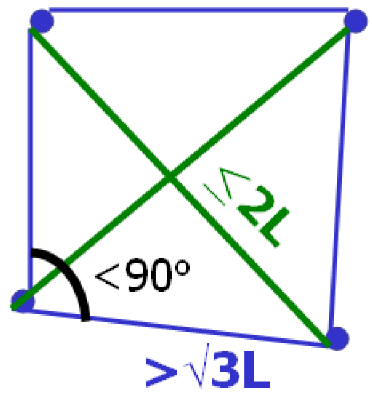


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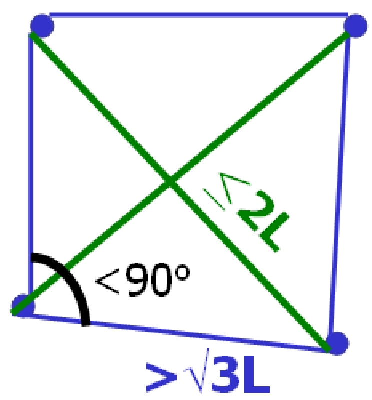
  - For d$\geq$3, G does not exclude any minor

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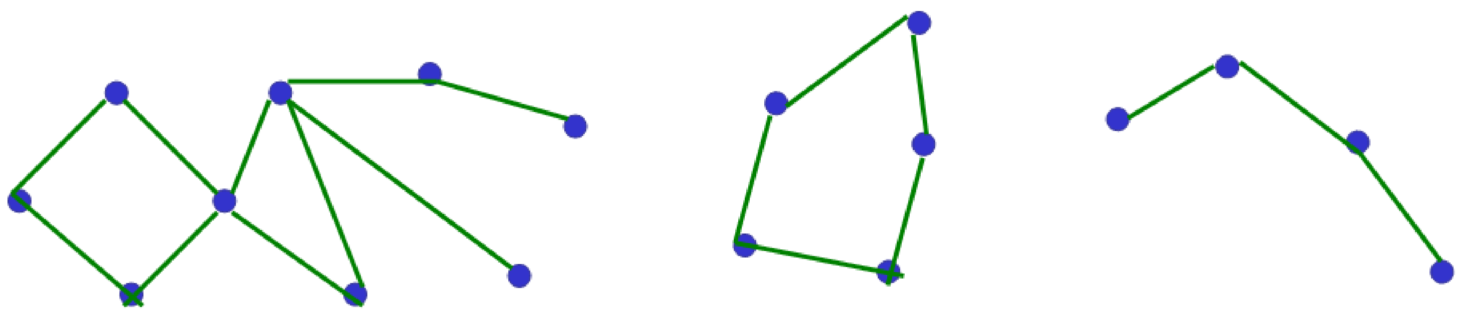

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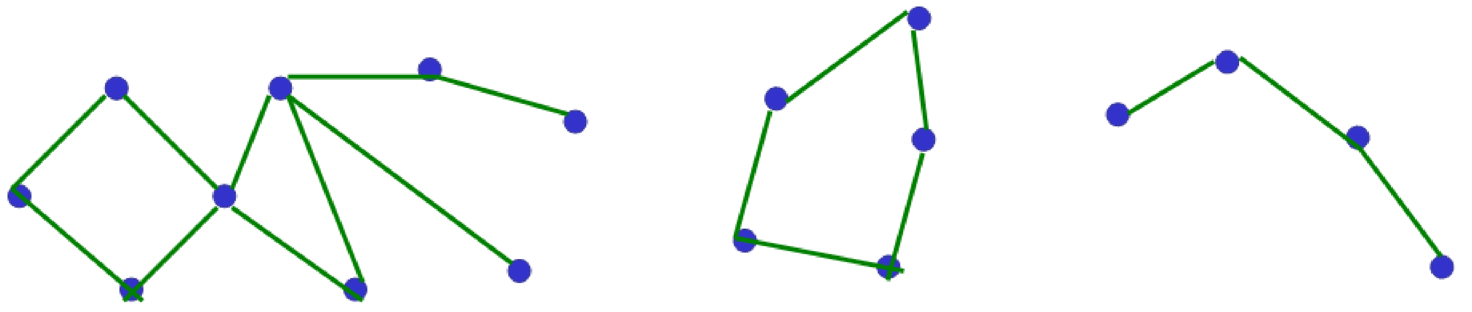

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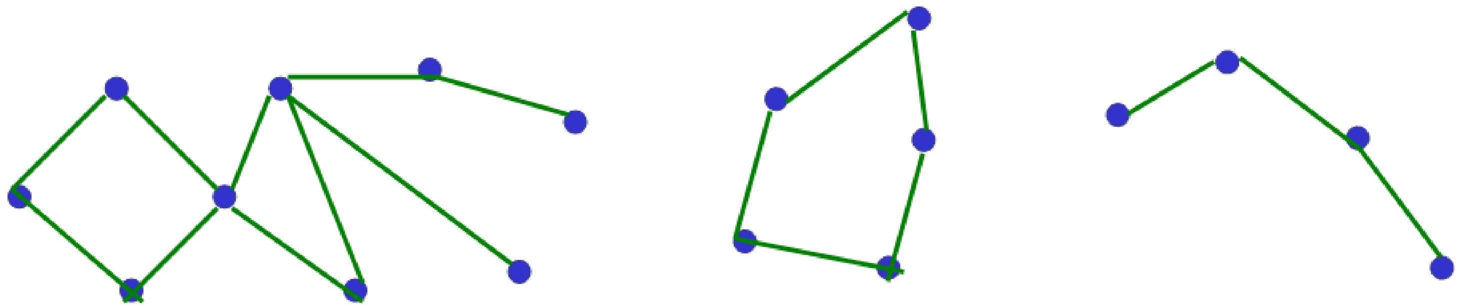
  - But time ~O(n) using ANN ☺



# Constructing G

- Graph G has
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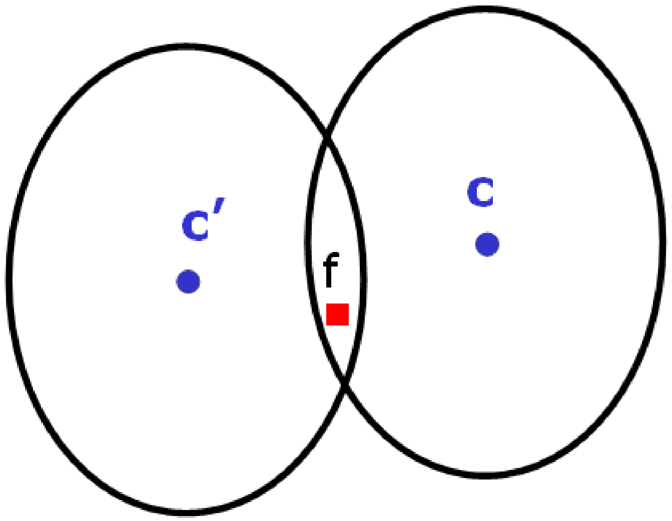


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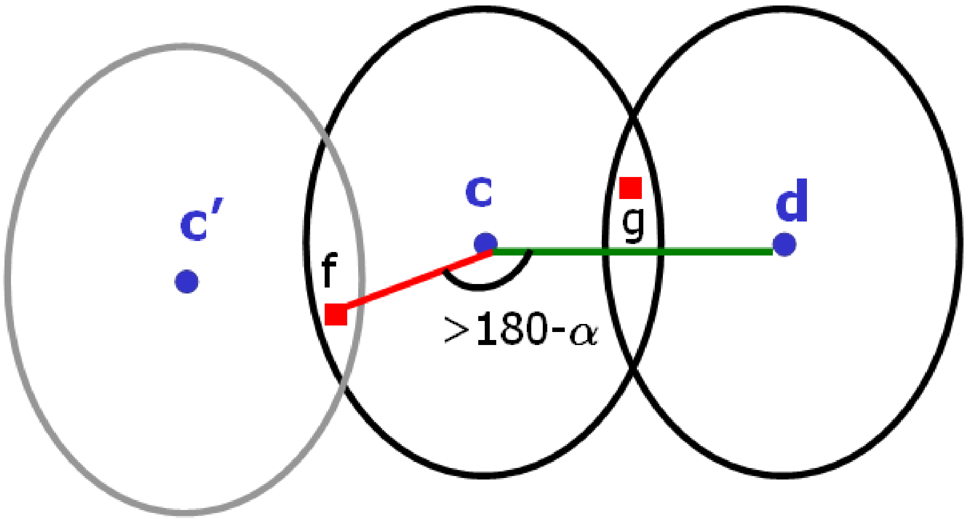
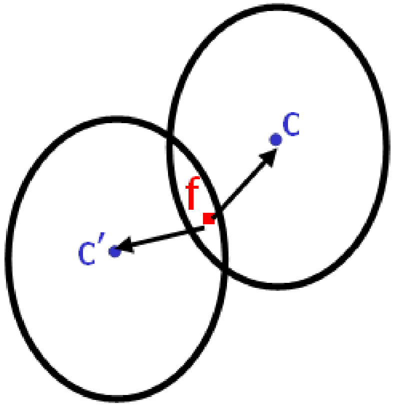



c′ and c ***fringe*** intersect

d ***antipode*** intersects <f,c>
(i) c & d fringe intersect
(ii) d is "too far" from f

# Constructing G

- Find next antipode intersection using ANN
  - Maintain pair of “end points”



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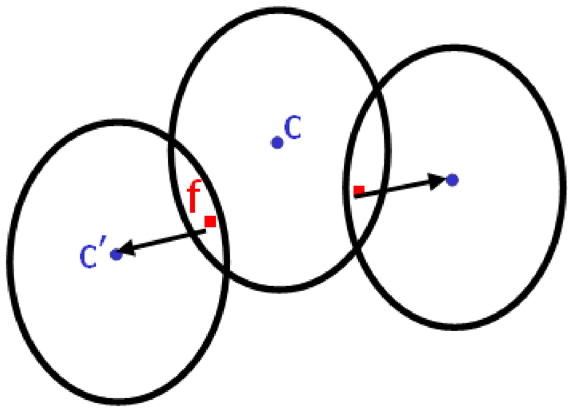


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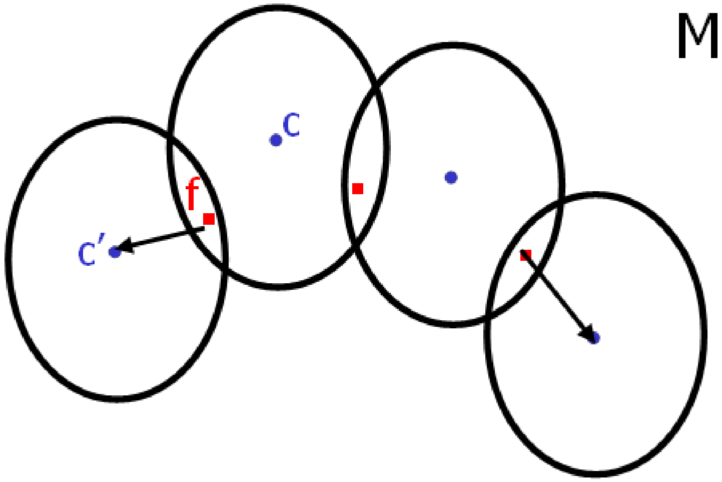

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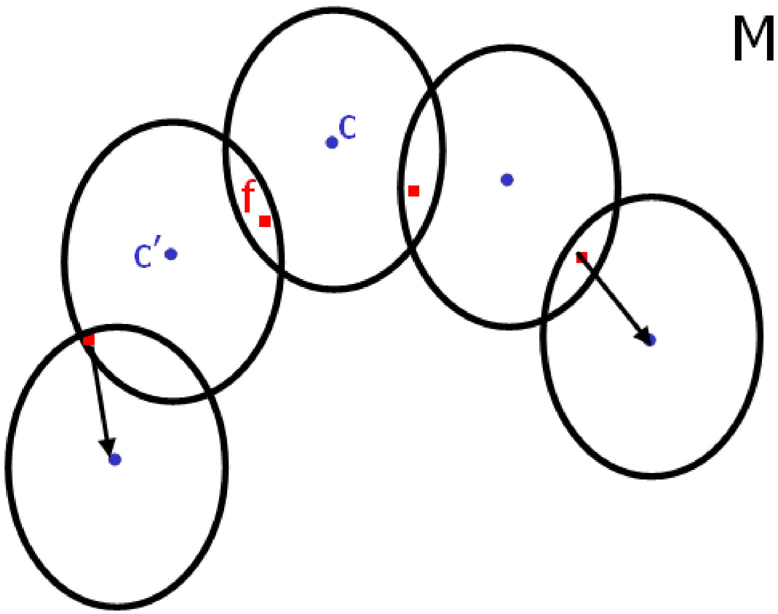

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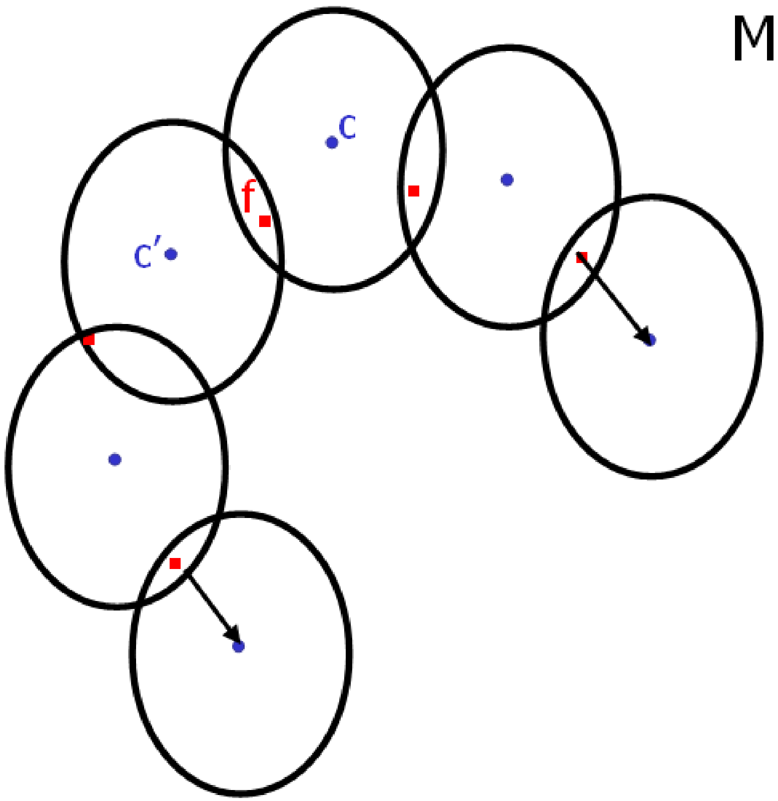

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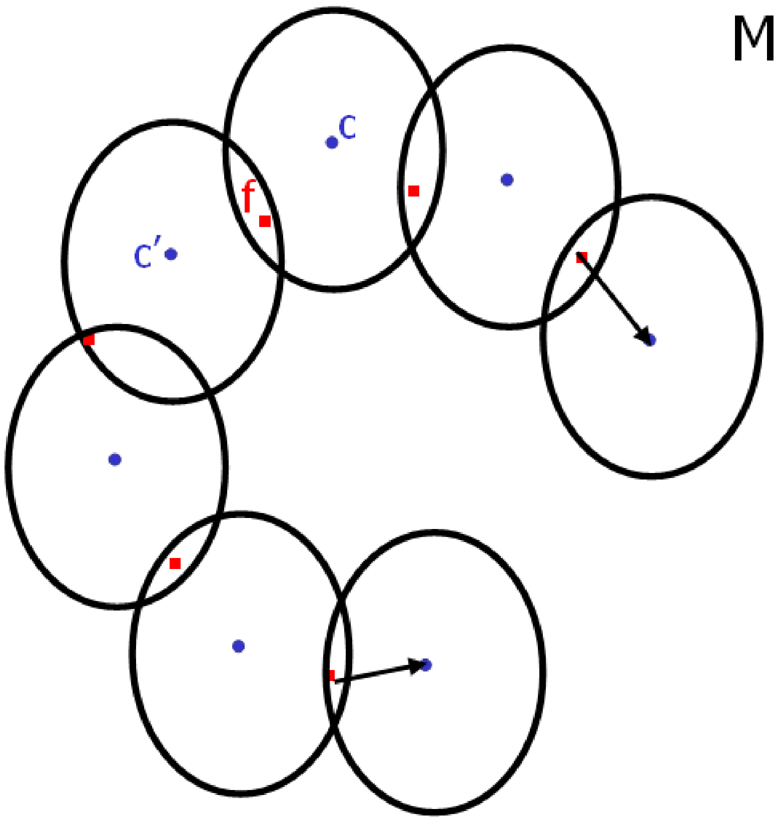

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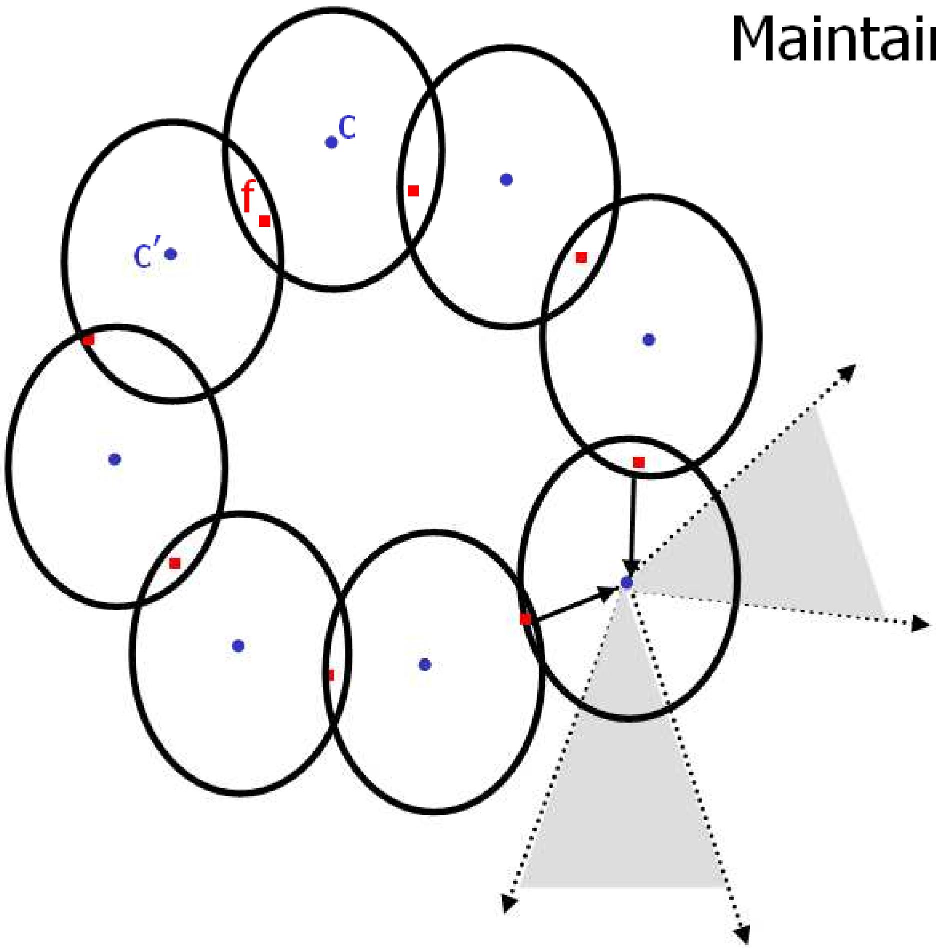

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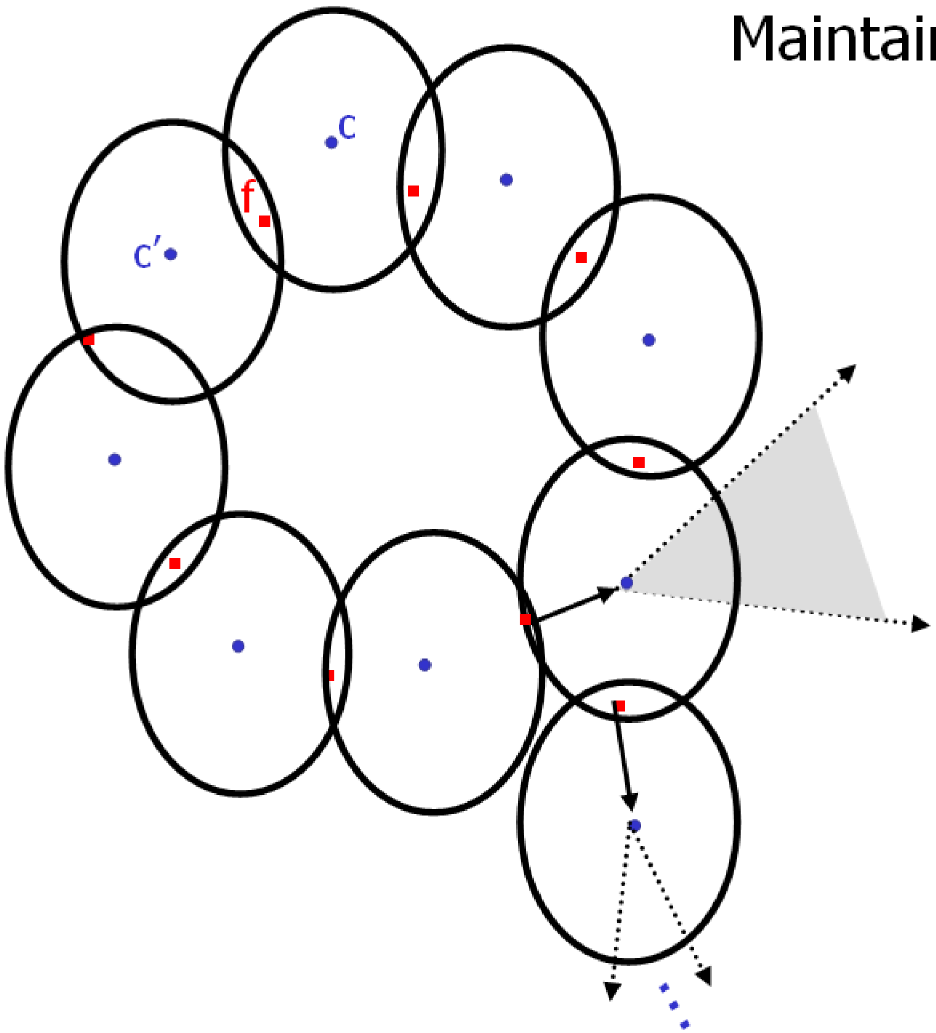

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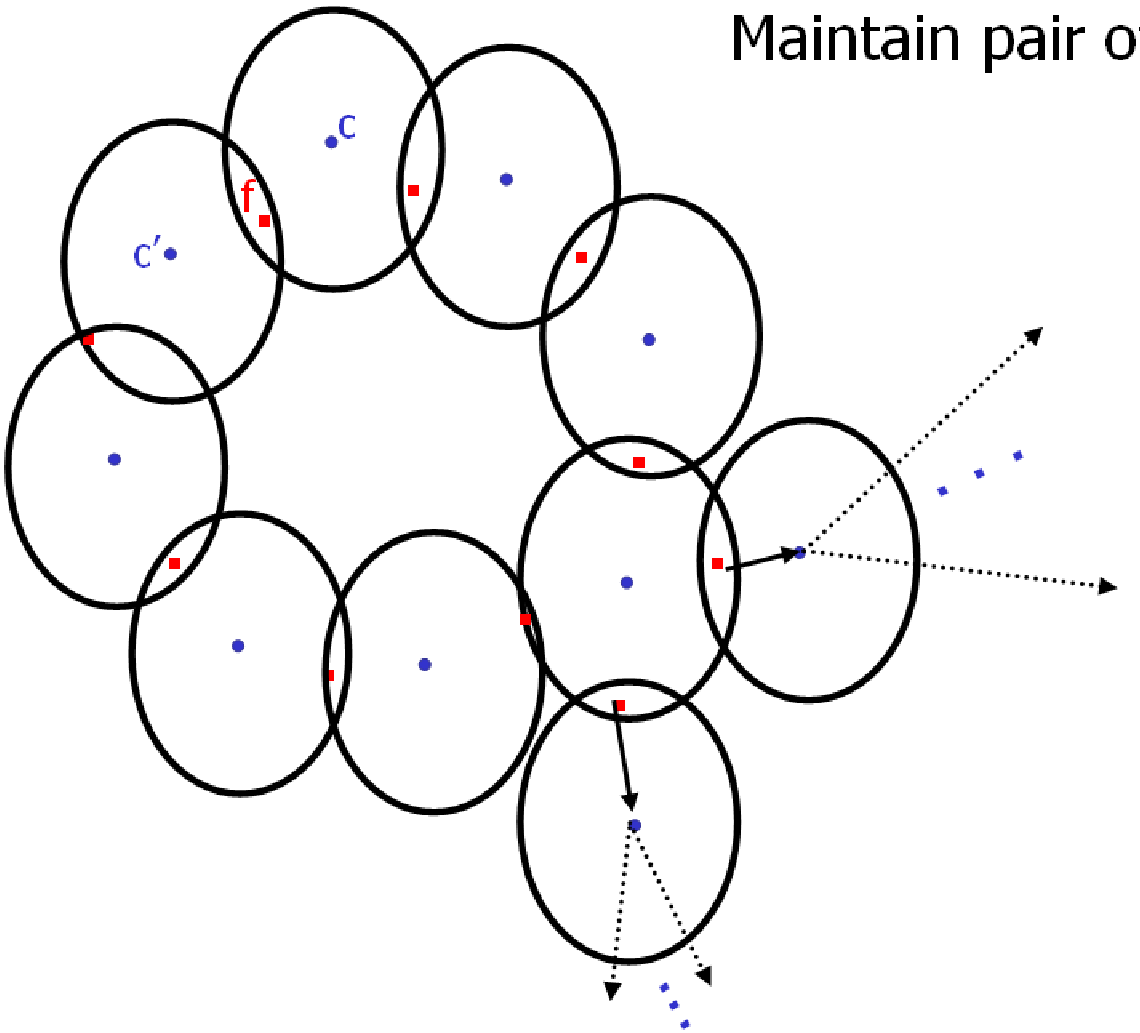

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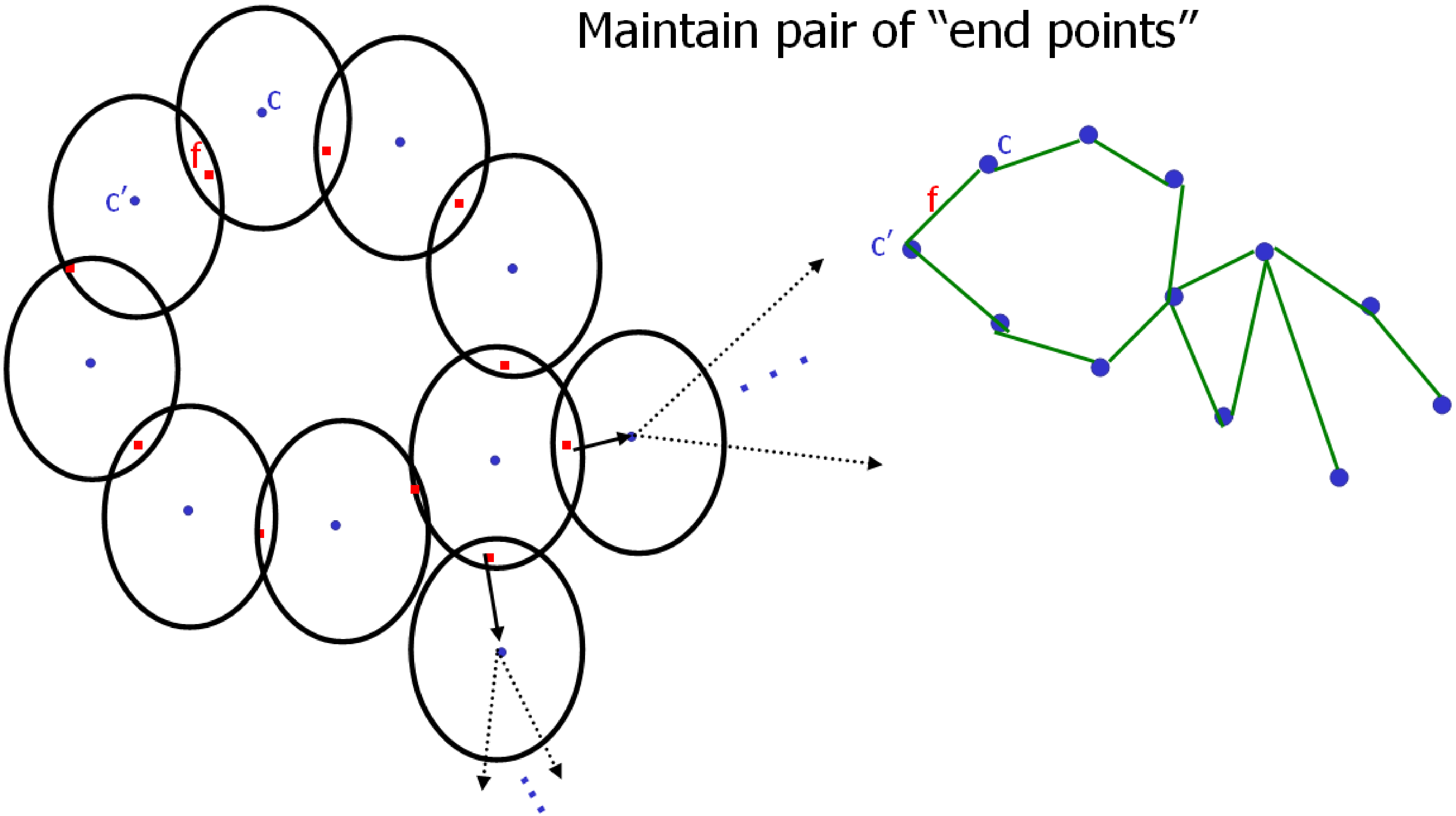

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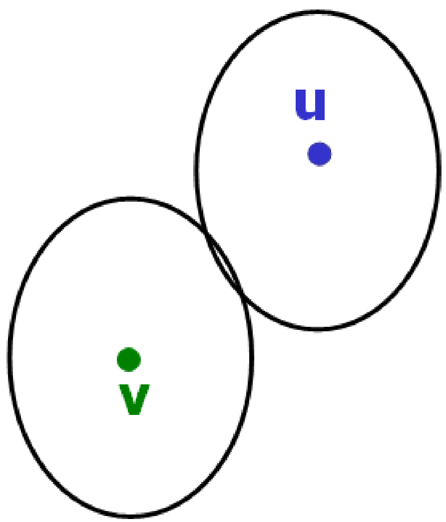

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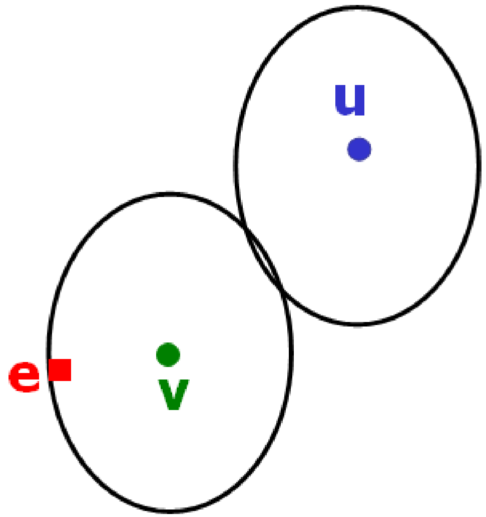

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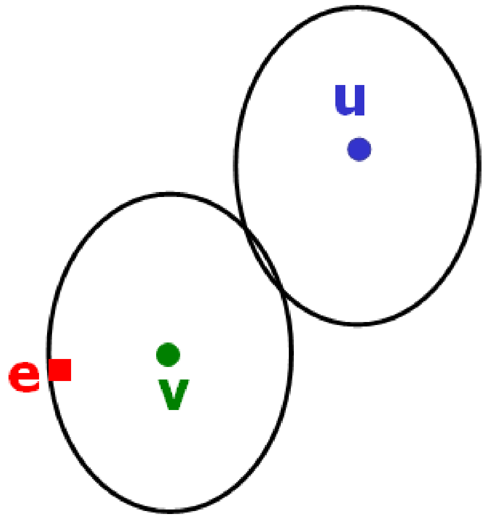

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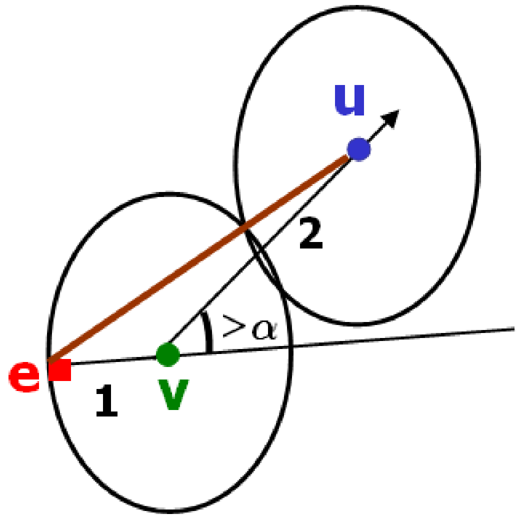

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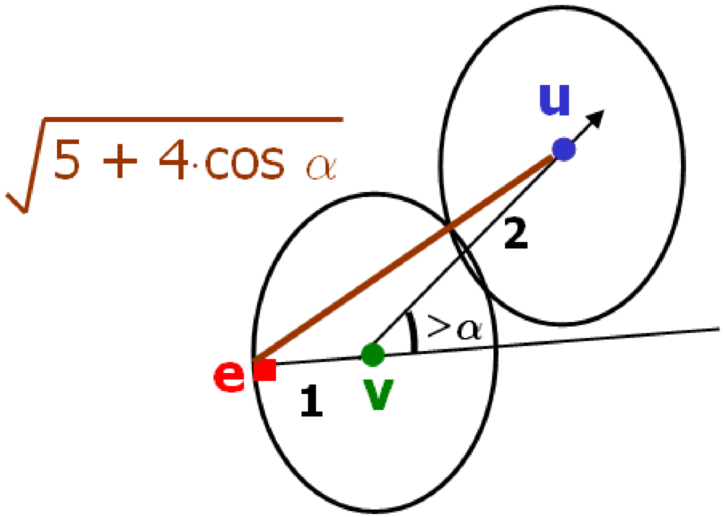

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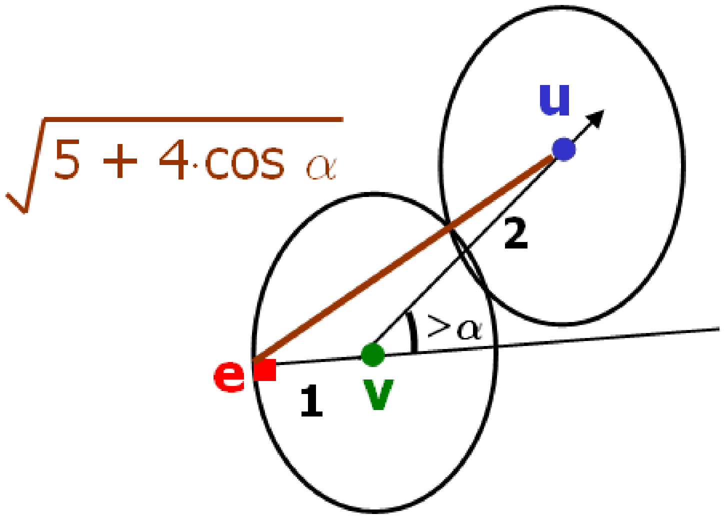


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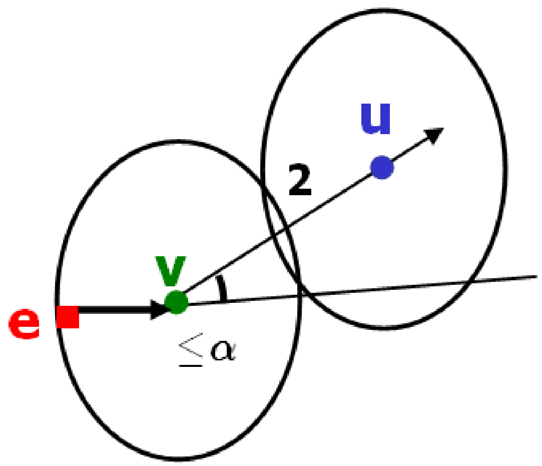

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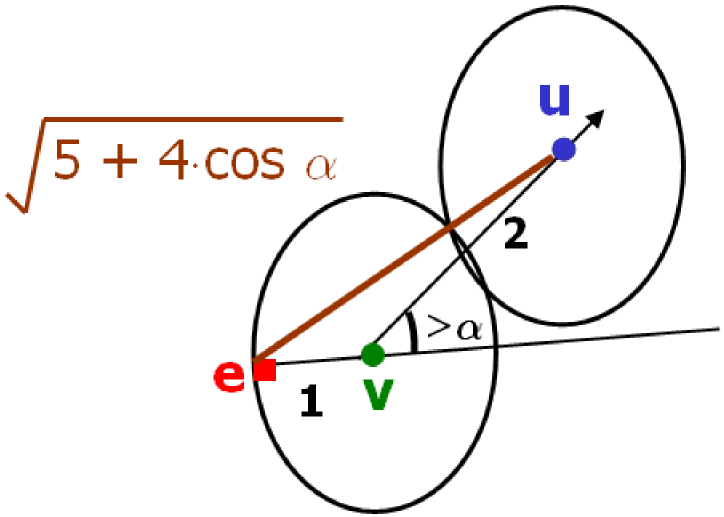


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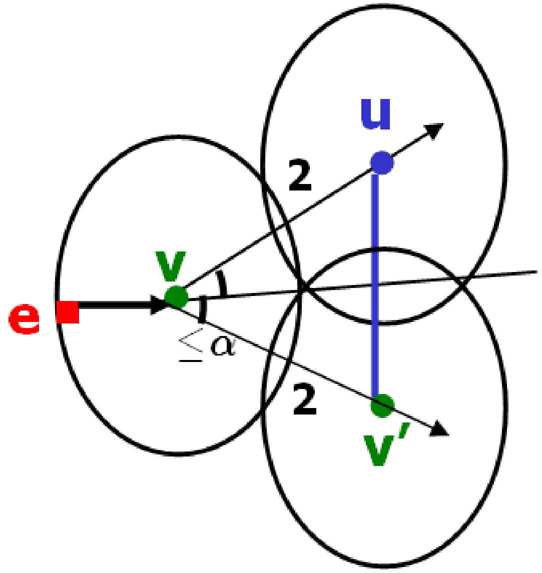

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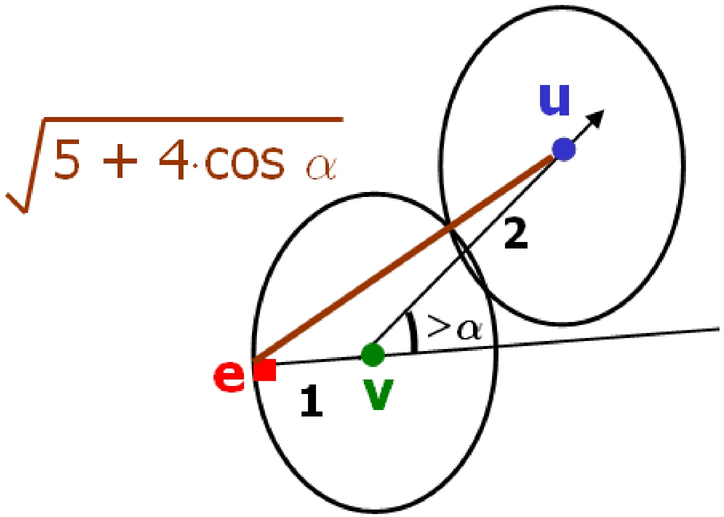


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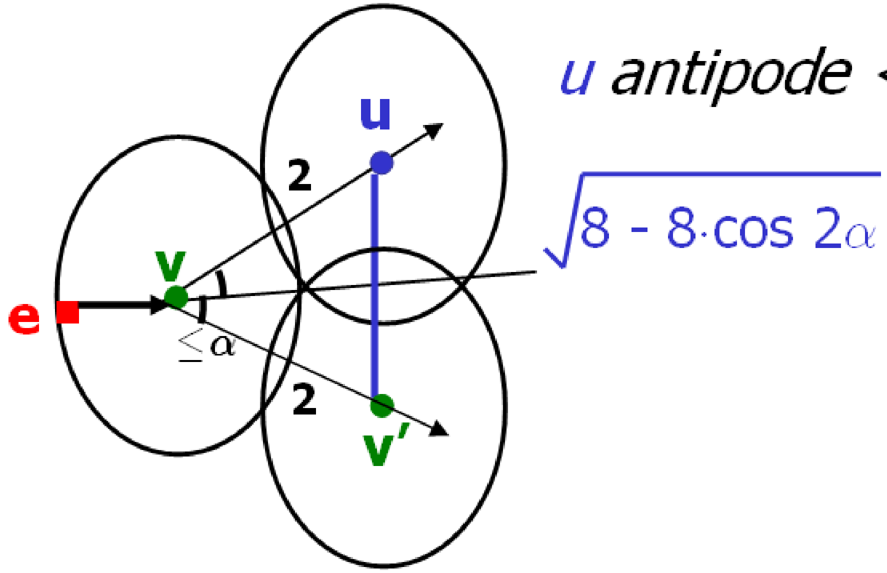

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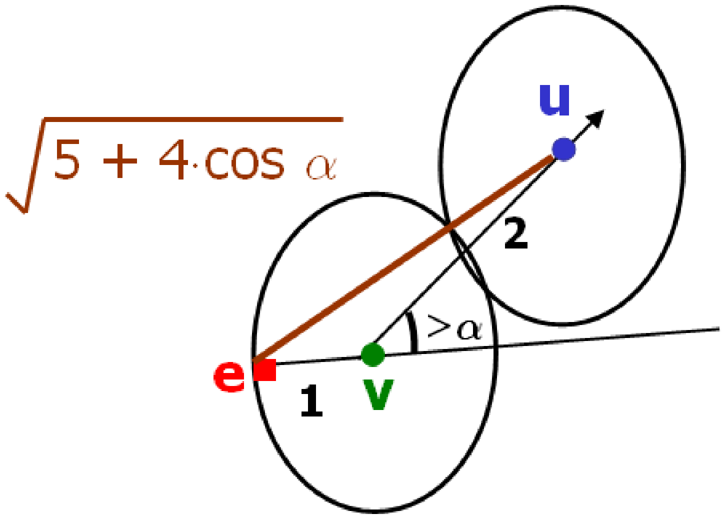


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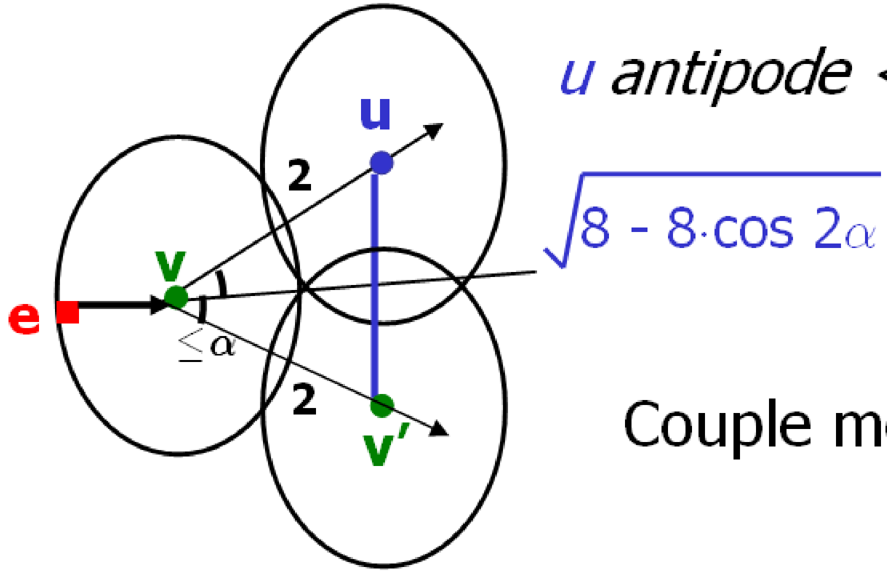


Couple more cases..

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