From Submodular to k-Submodular Maximization

Justin Ward

Joint Work with Stanislav Živný

July 31, 2013

Submodular Functions

Let U be a set of n elements.

Let $f : 2^U \to \mathbb{R}_{\geq 0}$ assign a value to each subset of U.

We say that f is *submodular* if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

Submodularity is also characterized by diminishing returns:

$$f(A + x) - f(A) \geq f(B + x) - f(B)$$

for all $A \subseteq B$ and $x \notin B$.

Bisubmodular Functions

We consider a generalization to biset functions $f : 3^U \to \mathbb{R}_{\geq 0}$.

Bisubmodular Functions

We consider a generalization to biset functions $f : 3^U \to \mathbb{R}_{\geq 0}$.

A biset function f is *bisubmodular* [Qi, 1988] if

$$f(A_1, A_2) + f(B_1, B_2) \geq f(A_1 \cap B_1,\, A_2 \cap B_2) + f(A_1 \cup B_1 \setminus (A_2 \cup B_2),\, A_2 \cup B_2 \setminus (A_1 \cup B_1))$$

Bisubmodular Functions

We consider a generalization to biset functions $f : 3^U \to \mathbb{R}_{\geq 0}$.

A biset function f is *bisubmodular* [Qi, 1988] if

$$f(A_1, A_2) + f(B_1, B_2) \geq f(A_1 \cap B_1,\ A_2 \cap B_2) + f(A_1 \cup B_1 \setminus (A_2 \cup B_2),\ A_2 \cup B_2 \setminus (A_1 \cup B_1))$$

Bisubmodular Functions

We consider a generalization to biset functions $f : 3^U \to \mathbb{R}_{\geq 0}$.

A biset function f is *bisubmodular* [Qi, 1988] if

$$f(A_1, A_2) + f(B_1, B_2) \geq f(A_1 \cap B_1,\ A_2 \cap B_2) + f(A_1 \cup B_1 \setminus (A_2 \cup B_2),\ A_2 \cup B_2 \setminus (A_1 \cup B_1))$$

Alternatively f is bisubmodular if and only if [Ando, Fujishige, Naitoh 1996]:

- The function $g(S) = f(S \cap A_1, S \cap A_2)$ is submodular for any partition (A_1, A_2) of U.

Bisubmodular Functions

We consider a generalization to biset functions $f : 3^U \to \mathbb{R}_{\geq 0}$.

A biset function f is *bisubmodular* [Qi, 1988] if

$$f(A_1,A_2) + f(B_1, B_2) \geq$$
$$f(A_1 \cap B_1,\ A_2 \cap B_2) + f(A_1 \cup B_1 \setminus (A_2 \cup B_2),\ A_2 \cup B_2 \setminus (A_1 \cup B_1))$$

Alternatively f is bisubmodular if and only if [Ando, Fujishige, Naitoh 1996]:

- The function $g(S) = f(S \cap A_1, S \cap A_2)$ is submodular for any partition (A_1, A_2) of U.
- $f(A_1 + e, A_2) + f(A_1, A_2 + e) \geq 2f(A_1, A_2)$

Bisubmodular Functions

We consider a generalization to biset functions $f : 3^U \to \mathbb{R}_{\geq 0}$.

A biset function f is *bisubmodular* [Qi, 1988] if

$$f(A_1,A_2) + f(B_1,B_2) \geq$$
$$f(A_1 \cap B_1,\ A_2 \cap B_2) + f(A_1 \cup B_1 \setminus (A_2 \cup B_2),\ A_2 \cup B_2 \setminus (A_1 \cup B_1))$$

Alternatively f is bisubmodular if and only if [Ando, Fujishige, Naitoh 1996]:

- The function $g(S) = f(S \cap A_1, S \cap A_2)$ is submodular for any partition (A_1, A_2) of U.
- $[f(A_1 + e, A_2) - f(A_1, A_2)] + [f(A_1, A_2 + e) - f(A_1, A_2)] \geq 0$

k-Submodular Functions

We can identify a solution $(S_1, \ldots, S_k)$ with a vector $\mathbf{v}$ in $\{0, 1, \ldots, k\}^U$, where $v_e = i$ iff $e \in S_i$, and $v_e = 0$ iff v_e in neither set. Then, let:

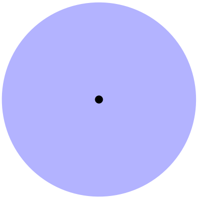
$$\min_0(s, t) = \begin{cases} 0, & s \neq 0, t \neq 0, s \neq t \\ \min(s, t), & \text{otherwise} \end{cases}$$

$$\max_0(s, t) = \begin{cases} 0, & s \neq 0, t \neq 0, s \neq t \\ \max(s, t), & \text{otherwise} \end{cases},$$

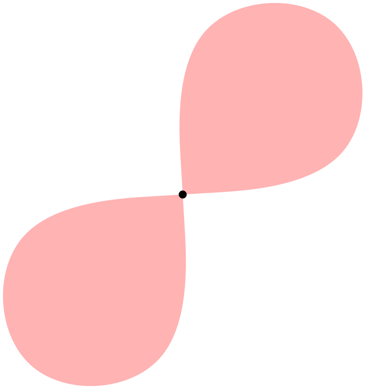
k-submodularity, is then

$$f(\mathbf{a}) + f(\mathbf{b}) \geq f(\min_0(\mathbf{a}, \mathbf{b})) + f(\max_0(\mathbf{a}, \mathbf{b}))$$

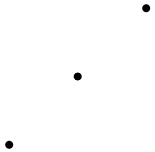
Application: Wireless Network Coverage



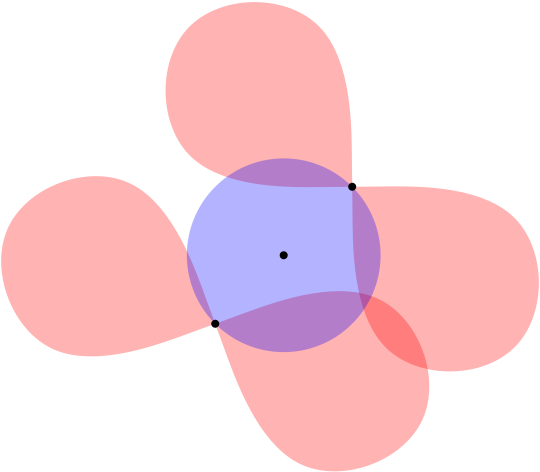
Application: Wireless Network Coverage



Application: Wireless Network Coverage



Application: Wireless Network Coverage



Related Work

Bisubmodular functions have many mathematical applications:

- Arose in the context of delta-matroids and pseudomatroids [Bouchet 87, Chandrasekaran and Kabadi 1988]
- Have recently been studied in the area of valued CSPs [Huber, Krokhin, and Powell 2013].
- (Strongly) polynomial algorithms for minimization [Fujishige, Iwata 2006; McCormick, Fujishige 2010].

Related Work

Bisubmodular functions have many mathematical applications:

- Arose in the context of delta-matroids and pseudomatroids [Bouchet 87, Chandrasekaran and Kabadi 1988]
- Have recently been studied in the area of valued CSPs [Huber, Krokhin, and Powell 2013].
- (Strongly) polynomial algorithms for minimization [Fujishige, Iwata 2006; McCormick, Fujishige 2010].

Recently, there has been some interest in maximization [Singh, Guillory, and Bilmes 2012].

- Give conditions under which bisubmodular maximization can be reduced to submodular.
- Use a single submodular function with a matroid constraint to enforce disjointness.
- Give examples of bisubmodular functions that require the related submodular function to be *negative*.

Randomized Greedy

Consider the following randomized algorithm (inspired by [Buchbinder, Feldman, Naor, Schwartz, 2012]):

- Set $S = (S_1, \ldots, S_k) = (\emptyset, \ldots, \emptyset)$
- For each $e \in U$:
  - Set $x_i = \max(0,\ f(S_1, \ldots, S_i + e, \ldots, S_k) - f(S)\,)$.
  - Set $\beta = \sum_{i=1}^{k} x_i$.
  - Add e to S_i with probability $\frac{x_i}{\beta}$

Theorem

Randomized Greedy is a $\dfrac{1}{\left(1 + \sqrt{\frac{k}{2}}\right)}$*-approximation algorithm for k-submodular maximization.*

Proof Sketch

Let $S^{(j)} = (S_1^{(j)}, \ldots, S_k^{(j)})$ be the current solution after j elements have been considered.

Suppose we move these j elements in the optimal solution so they agree $S^{(j)}$ and call the result $O^{(j)}$.

Then, $O^{(0)}$ is the optimal solution, and $O^{(n)} = S^{(n)}$ is the greedy solution.

It suffices to bound the expected decrease in $O^{(j)}$ at each phase.

Proof Sketch

Consider the $(j+1)$th element e, and suppose that $e \in O_p$.

Remove e from $O^{(j)}$ and call the result A.

- As in the algorithm,

$$x_i = \max(0, f(S_1, \ldots, S_i + e, \ldots, S_k) - f(S))$$

- Also, define

$$a_i = f(A_1, \ldots, A_i + e, \ldots, A_k) - f(A)$$

Then, from bisubmodularity, we have:

- $a_i \leq x_i$ for all $1 \leq i \leq k$.
- $\sum_{i=1}^{k} a_i \geq 0$.

Proof Sketch

$$\mathbb{E}[f(O^{(j)}) - f(O^{(j+1)})] = \frac{1}{\beta}\sum_{i=1}^{k} x_i(a_p - a_i)$$

$$\mathbb{E}[f(S^{(j+1)}) - f(S^{(j)})] = \frac{1}{\beta}\sum_{i=1}^{k} x_i^2$$

We show that $\sum_{i=1}^{k} x_i(a_i - a_p) \leq \sqrt{\frac{k}{2}\sum_{i=1}^{k} x_i^2}$

Main idea: consider extreme point solutions of the LP

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{k} x_i(a_p - a_i) \\ \text{subject to} \quad & a_i \leq x_i \qquad\qquad 1 \leq i \leq k \\ & \sum_{i=1}^{k} a_i \geq 0 \end{aligned}$$

Proof Sketch

Summing up the expected losses over all phases

$$f(O^{(0)}) - f(O^{(n)}) \leq \sqrt{\frac{k}{2}}\left[f(S^{(n)}) - f(S^{(0)})\right]$$

$$f(O) - f(S) \leq \sqrt{\frac{k}{2}} f(S)$$

Embedding Submodular Functions

Let $g : 2^U \to \mathbb{R}_{\geq 0}$ be a submodular function. [Fujishige, Iwata, 2005] consider the following embedding:

$$f(A, B) = g(A) + g(U \setminus B)$$

Note that $f(A, U \setminus A) = 2g(A)$.

Recall: $[f(A + x, B) - f(A, B)] + [f(A, B + x) - f(A, B)] \geq 0$.

So, we can extend any solution to a partition without any loss in f. Thus, f preserves approximation, and so our randomized greedy 1/2-approximation is tight.

Embedding Submodular Functions

What does the randomized greedy algorithm look like on this embedding?

$$f(A, B) = g(A) + g(U \setminus B)$$

- We maintain 2 solutions $X = A$ and $Y = U \setminus B$.
- Initially, we $X = \emptyset$ and $Y = U$.
- At each step we either add e to X or remove e from Y with probability proportional to the increase in g.

Exactly the algorithm of [Buchbinder, Feldman, Naor, Schwarz, 2012].

Summary

$k =$	1	2	≥ 3
Deterministic Greedy	$1/3$	$1/3$	$1/(k+1)$
Random Greedy	$1/2$	$1/2$	$1/(1+\sqrt{k/2})$
Naive Random	$1/4$	$1/4$	$1/k$

Open Questions

- Are our results tight for $k \geq 3$?
- Can we attain better approximations for specific cases (like the monotone case).
- What about constrained optimization (budget, knapsack, matroid, etc.)?
- What about other variants, such as skew bisubmodularity?
- Can we find any embedding of submodular or bisubmodular into k-submodular for $k \geq 3$?

Open Questions

- Are our results tight for $k \geq 3$?
- Can we attain better approximations for specific cases (like the monotone case).
- What about constrained optimization (budget, knapsack, matroid, etc.)?
- What about other variants, such as skew bisubmodularity?
- Can we find any embedding of submodular or bisubmodular into k-submodular for $k \geq 3$?

Thank You