The Directed Steiner Tree Polyhedron

Ahmad Abdi

Dept. of C&O, UWaterloo

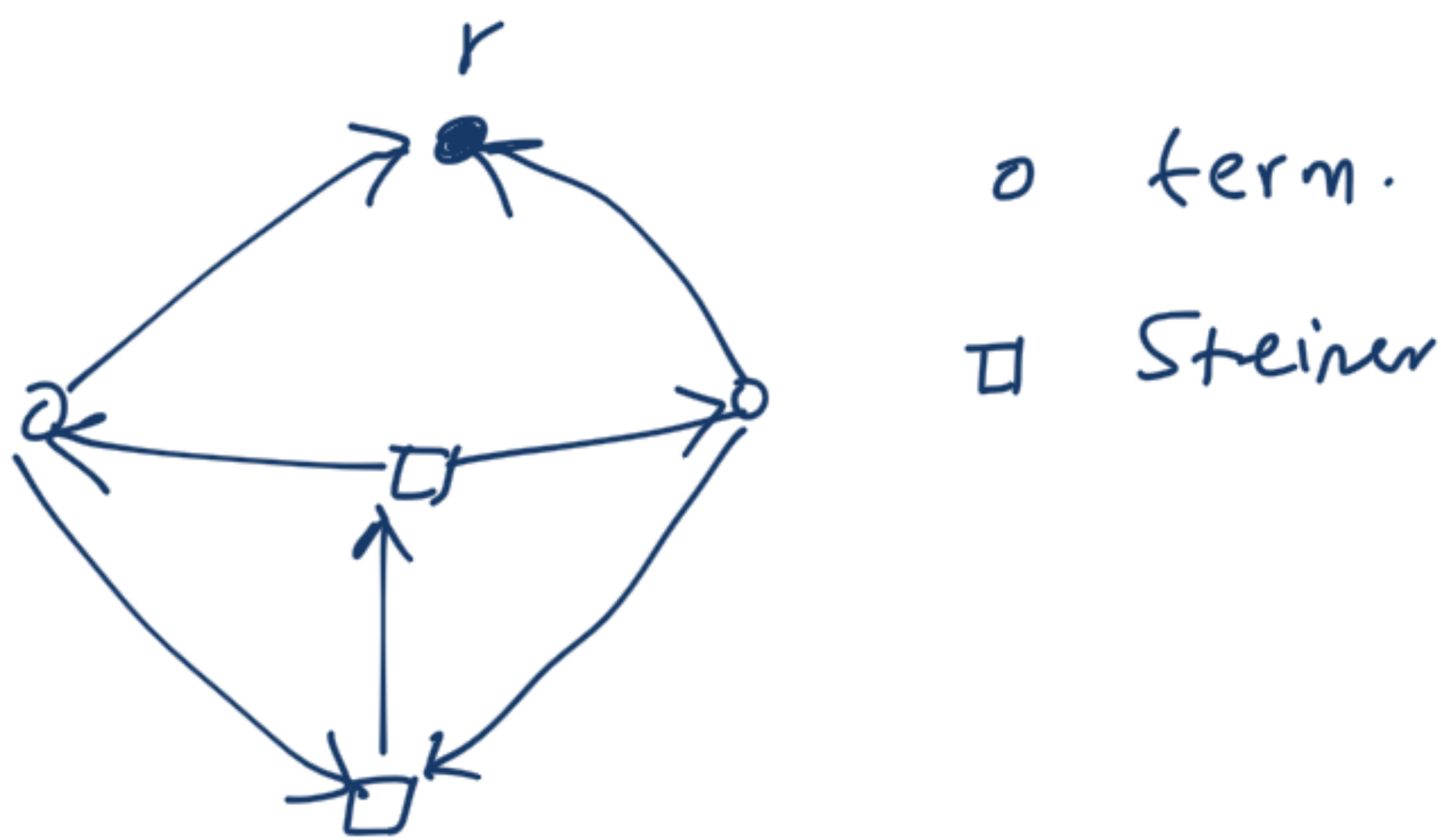
Joint work with

Andreas Feldmann
Bertrand Guenin
Jochen Könemann
Laura Sanità

Directed Steiner Tree problem

- $D = (V, A)$,
- $V = \underbrace{\text{terminals}}_{R} \dot\cup \text{ Steiner nodes}$
- root $r \in R$,
- $w \in \mathbb{R}_+^A$.

Objective: find a minimum weight r-arborescence spanning R.



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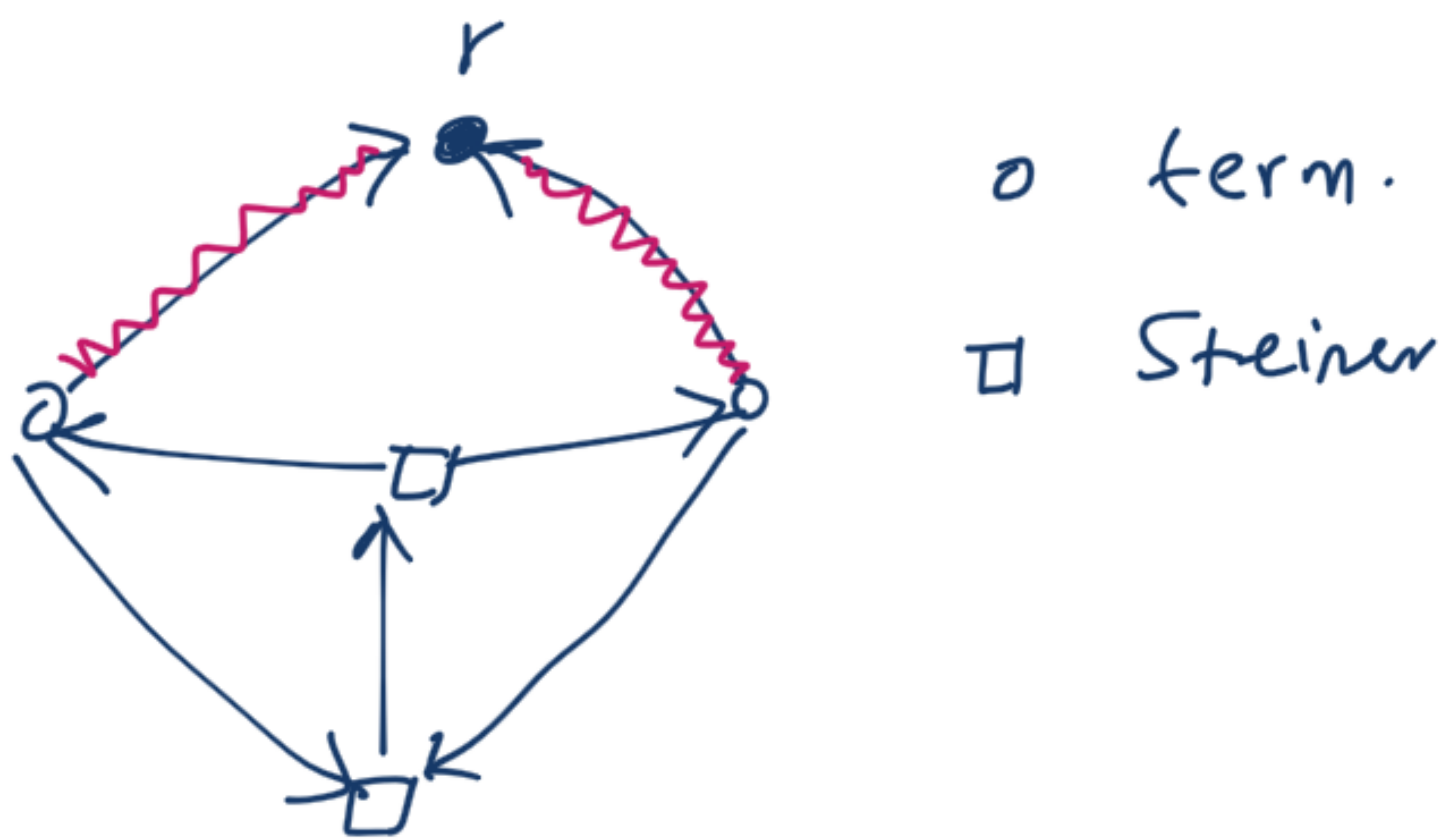

This is NP-hard...

An IP interpretation:

$$
\begin{aligned}
\min \quad & w^T x \\
\text{s.t.} \quad & x(\delta^+(U)) \geq 1, \\
& \quad \forall U \subset V \setminus r,\ U \cap R \neq \emptyset. \\
& x \in \{0,1\}^A
\end{aligned}
$$

An LP relaxation:

$$\begin{aligned} \min \quad & w^T x \\ \text{s.t.} \quad & x(\delta^+(U)) \geq 1, \\ & \quad \forall U \subset V \setminus r,\ U \cap R \neq \emptyset. \\ & x \geq 0 \end{aligned}$$

Steiner Tree polyhedron:

$$P = \{ x \in \mathbb{R}^A :$$

$$x(\delta^+(U)) \geq 0,$$

$$\forall U \subset V \setminus r,\ U \cap R \neq \emptyset$$

$$x \geq 0 \}$$

Obs.: If P is integral then the problem is easy.

Q: When is P integral?

Known results: P is integral when

- $|R| = 2$ (shortest directed path)
- $R = V$ (Edmonds '67)
- D is series-parallel
  (Prodons et al. '85, Goemans '94, Schaffers '91)

Common framework to prove all results with shorter proofs, and get new results:

Via minor operations ...

$$P = \{ x \geq 0 : Ax \geq 1 \}$$

0,1 matrix

Two minor operations: $e \in A$

- $P \cap \{ x : x_e = 0 \}$ - deletion
- $P \cap \{ x : x_e = 1 \}$ — contraction

Lehman '90:

If P is non-integral then it has a highly regular non-integral minor.

If P is non-integral, can we use this result to find

- 3 terminals,
- 1 Steiner vertex,
- a subdivision of K_4,

and

- a directed cycle or 3 Steiner vertices, etc.

So P is integral if

D is acyclic & has ≤ 2

Steiner vertices.

A conjecture:

If P is non-integral then there is a subdivision of K_4 containing 3 terminals and a Steiner vertex.

Lehman's Result on non-integral set covering polyhedra is a useful & powerful tool.

Thanks!