Adaptable colouring and colour critical graphs

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Adapted k-colouring of graphs

Definitions. A graph G is adaptably k-colourable if for every k-edge colouring c', there is a k-vertex colouring c such that for every edge xy in G, not all of $c(x)$, $c(y)$, and $c'(xy)$ are the same.

The edge xy is monochromatic if $c(x)=c(y)=c'(xy)$.

The adaptable chromatic number of G, $\chi a(G)$, is the least k such that G is adaptably k-colourable.

Adapted *k*-colouring as a game

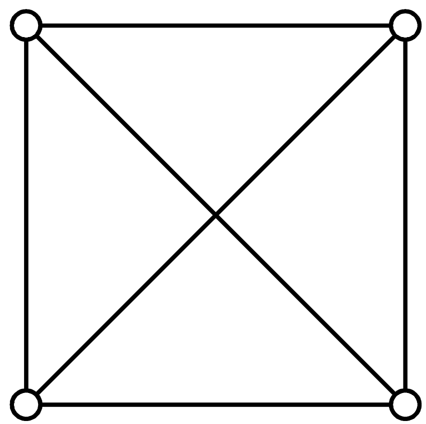
- There are two players E and V.
- Player E colours the edges of a graph G first using colours in $\{1,2,\ldots,k\}$.
- Player V then colours vertices of G using the same set of colours.
- Player V wins if he can colour the vertices without creating any monochromatic edges.
- Otherwise E wins.

Adapted *k*-colouring as a game

- The least number of colours that player V always has a winning strategy is the adaptable chromatic number of G, $\chi a(G)$.

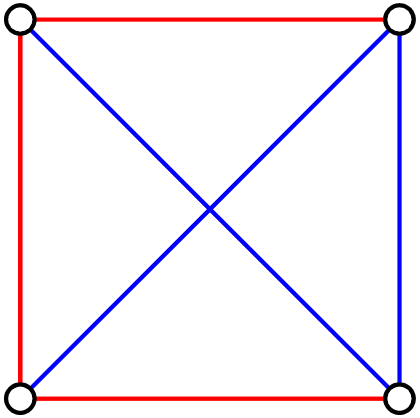
Example. $K4$

- Consider the graph $K4$:



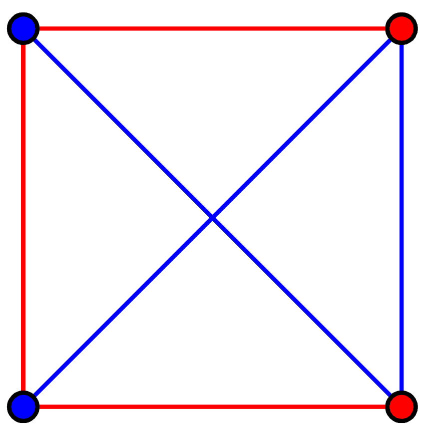
A 2-edge colouring of *K*4.

- E colours the edges in two colours:



An adapted 2-colouring

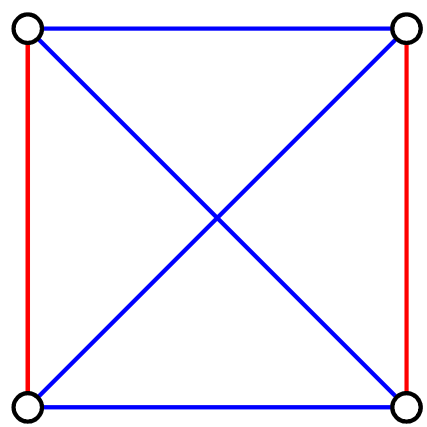
- V colours the vertices in two colours:



There is no monochromatic edge.

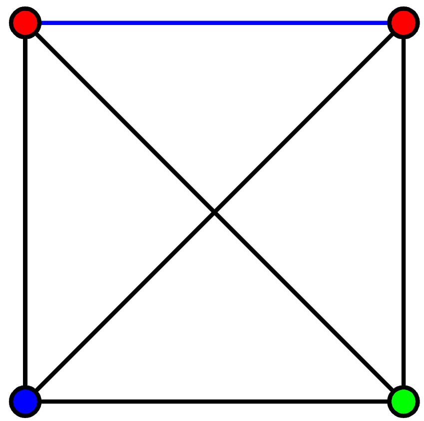
A winning strategy for E with 2 colours

- E has a winning strategy with two colours:



Therefore $\chi a(K4) > 2$.

A winning strategy of V with 3 colours



$\chi a(K4) = 3.$

Colour critical graphs

- A graph G is k-critical if $\chi(G) = k$ and $\chi(G$ □ $e) = k - 1$ for every edge e in G.
- A k-critical graph can be coloured with $k - 1$ colours such that there is only one edge joining two vertices of the same colour.

Fact. If G is k-critical then $\chi a(G) \leq k$ □ 1.

Problem. (Molloy and Thron 2012)

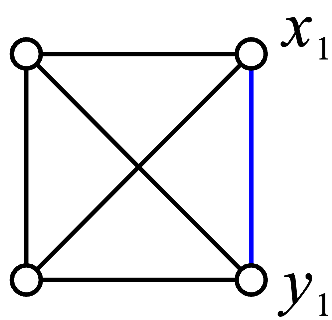
Are there any critical graphs G with

$\chi a(G) = \chi(G)$ □ 1?

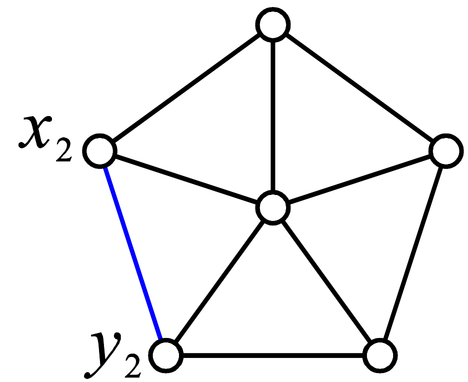
Construction 1

The Hajós' construction.

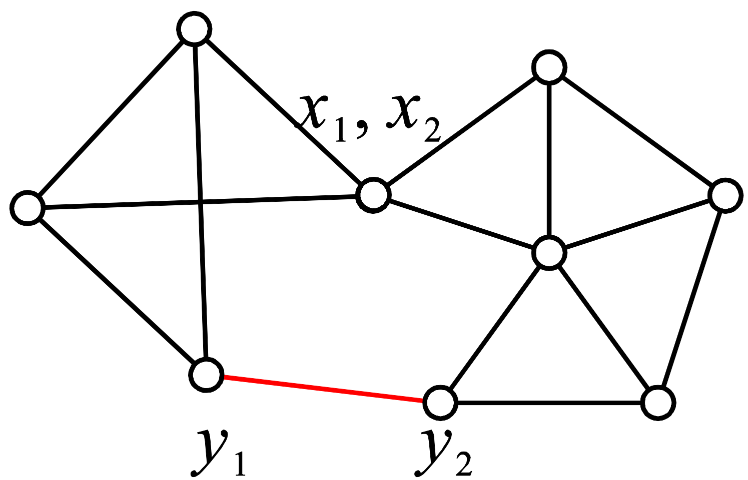
$G_1 =$



$G_2 =$



The result is



Construction 1

Let G be the graph obtained by applying the Hajós' construction to two graphs G_1 and G_2.

Fact. If both G_1 and G_2 are k-critical, then G is also k-critical.

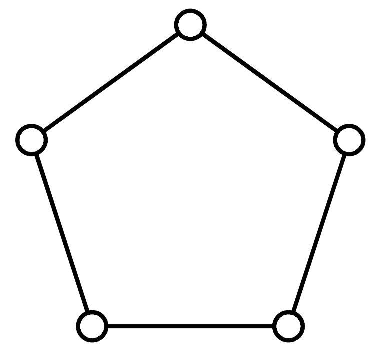
Fact. (Huizenga 2008) If $\chi_a(G_1) \geq k$ and $\chi_a(G_2) \geq k$, then $\chi_a(G) \geq k$.

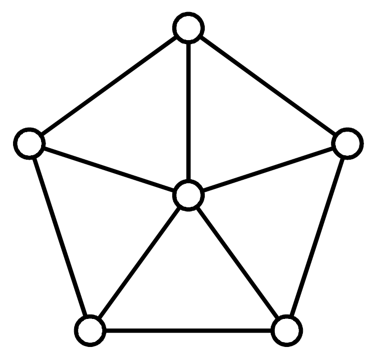
Implication. If there is a k-critical graph G with $\chi_a(G) = k$ □ 1 then there are infinitely many such graphs.

Construction 2

$G_1 \vee G_2$, the join of G_1 and G_2

$G_1 =$ 

$G_2 =$ 

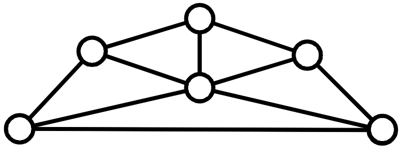
$G_1 \vee G_2 =$  $= W_5$

Construction 2

- If G_1 is a k_1-critical graph and G_2 is a k_2-critical graph, then $G_1 \vee G_2$ is a (k_1+k_2)-critical graph.
- However, it can happen that

$$\chi_a(G) < \chi_a(G_1) + \chi_a(G_2).$$

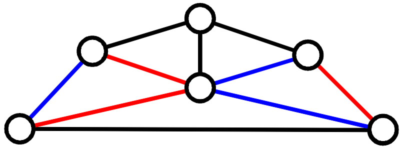
The graph $W5$



$W5$

$W5$ is 4-critical.

$\chi a(W5) \geq 3$.

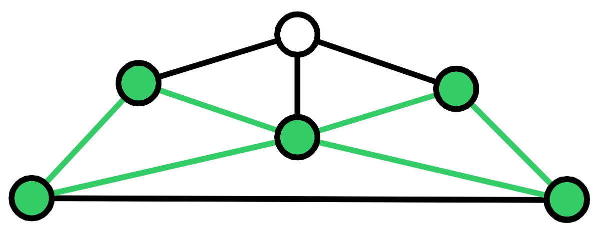


Therefore, $\chi a(W5) = 3$.

An important property of *W*5

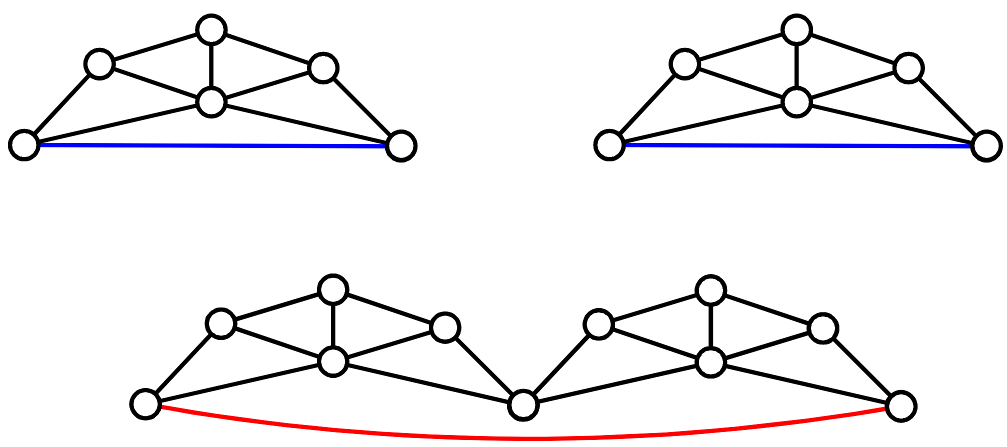
*W*5 has a proper subgraph *H*4 such that

$$\chi a(H4) = 3.$$



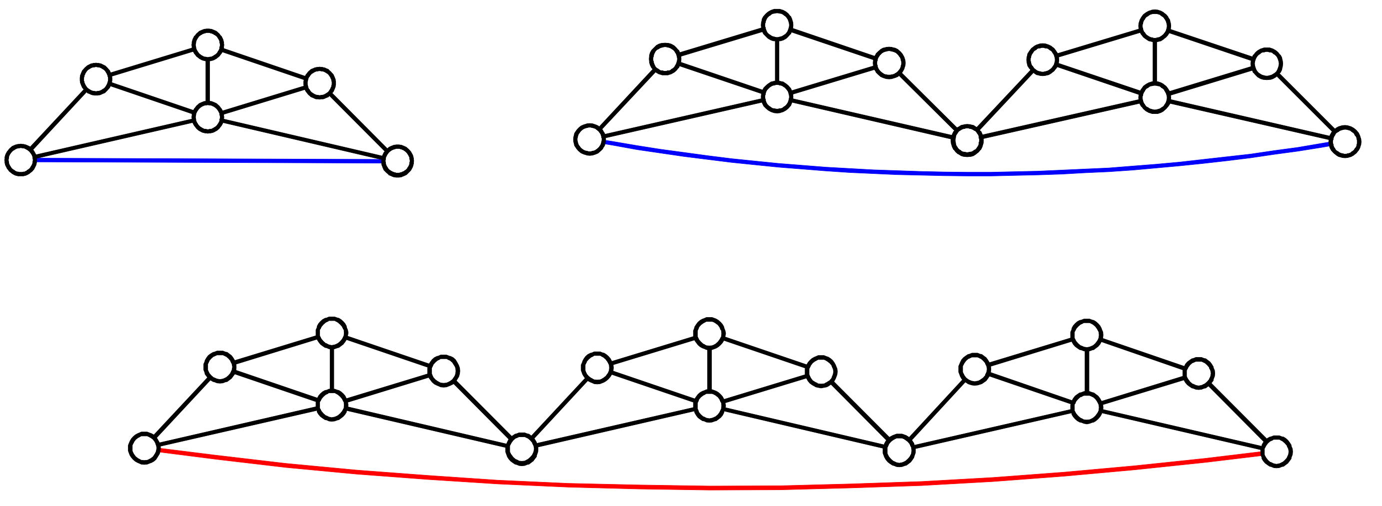
The construction for $k = 5$. (1)

We apply Hajós' construction to two copies of W_5.



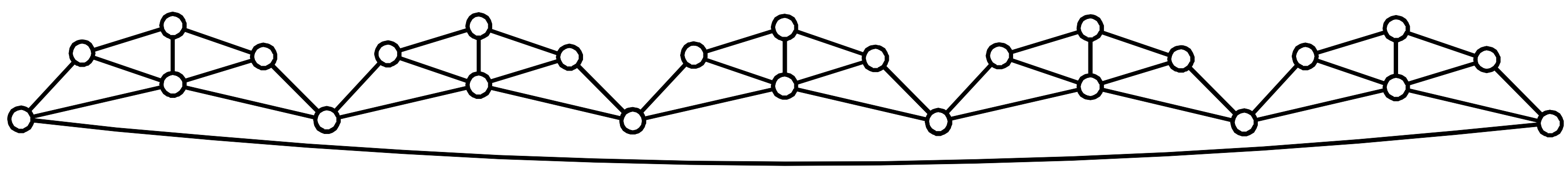
The construction for $k = 5$. (2)

We apply Hajós' construction one more time.



The construction for $k = 5$. (3)

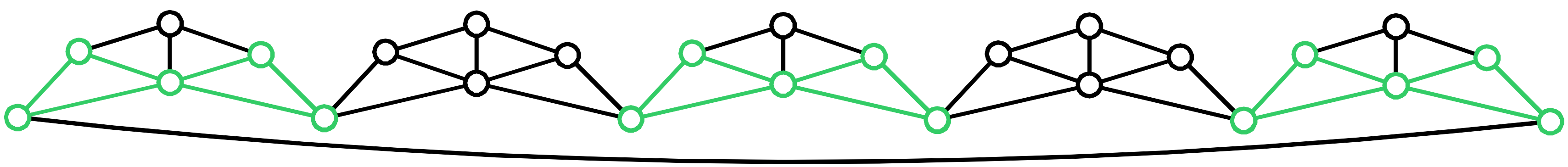
We continue applying Hajós' construction to get this graph $F4$.



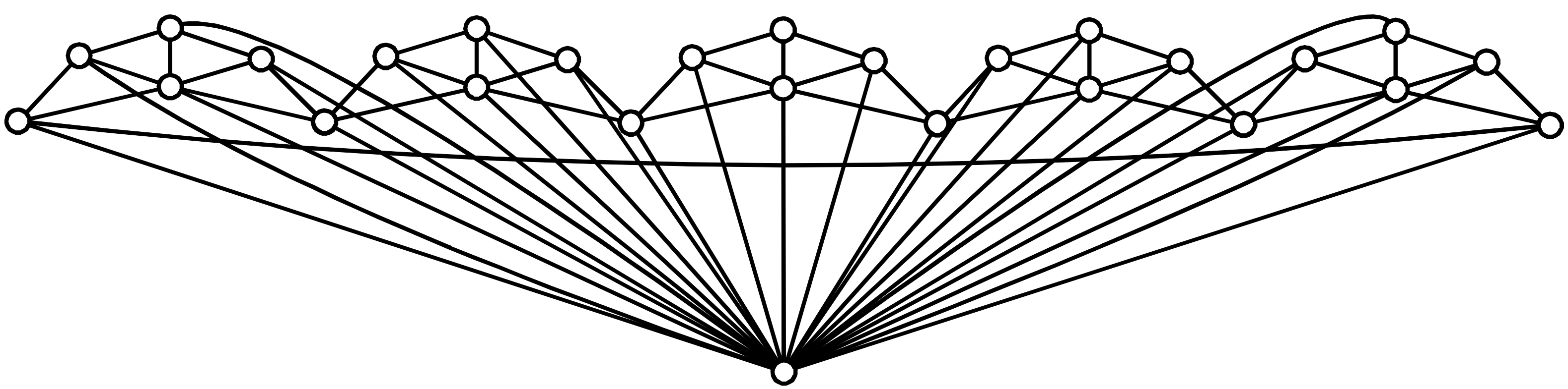
$F4$ is 4-critical.

The construction for $k = 5$. (4)

$F4$ contains three disjoint copies of $H4$.



$G_5 = K_1 \vee F_4$.

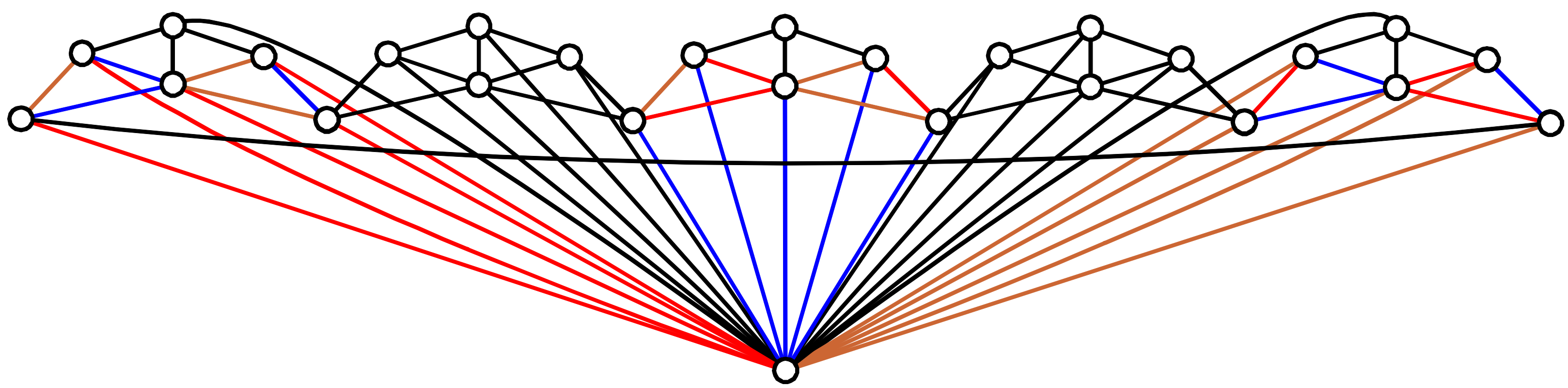


The construction for $k = 5$. (5)

$G5$ is 5-critical. Therefore $\chi a(G5) \leq 4$.

Claim. $\chi a(G5) \geq 4$.

We show that Player E has a winning strategy with 3 colours on $G5$.



General case

Theorem. For every integer k such that $k \geq 4$, there is a k-critical graph G_k that contains a proper subgraph H_k such that

$$\chi_a(H_k) \geq k \,\square\, 1.$$

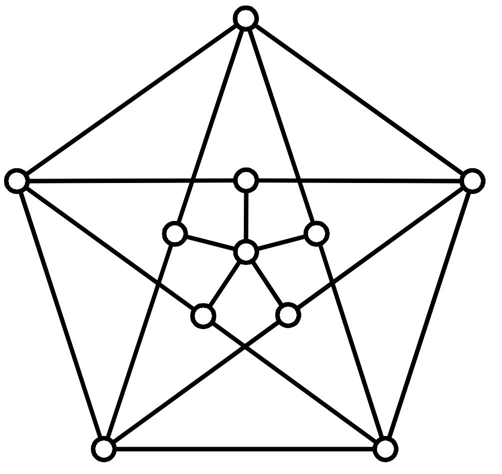
$K4$ again.

- $\chi(K4) = 4$ and $\chi(K4 \square e) = 3$ for every edge e in $K4$.
- $\chi a(K4) = 3$.
- $\chi a\ (K4 \square e) = 2$ for every edge e in $K4$.

Question. Are there any other such “double critical” graphs G with $\chi a = \chi(G) \square 1$?

The Grötzsch graph

Let G be the Grötzsch graph.



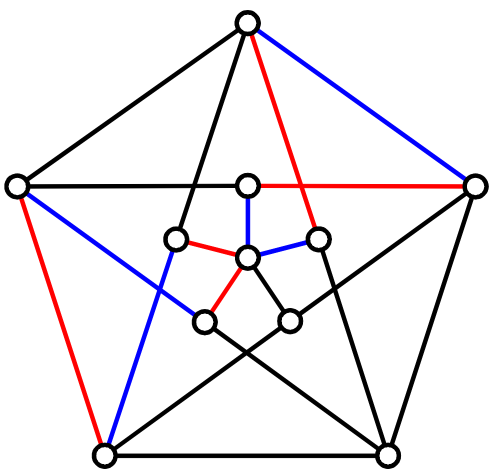
Fact. G is 4-critical.

Fact. G is triangle-free.

The Grötzsch graph

Fact. $\chi a(G) = 3$.

Player E has a winning strategy if there are two colours.



Fact. There are triangle-free 4-critical graphs with adaptable chromatic number 3.

More questions

- Question 1: Are there triangle-free k-critical graphs with adaptable chromatic number k-1 for every $k \geq 5$?
- Question 2: Are there k-critical graphs with adaptable chromatic number k-1 and girth g for every $k \geq 4$ and $g \geq 4$?

Lower bound

- (Greene, 2004)

$$\chi_a(G) \geq \frac{\chi(G)}{\sqrt{n \log(\chi(G))}}$$

where n is the number of vertices in G.

Conjecture. (Greene) There is a function f such that $\chi_a(G) \geq f(\chi(G))$ and $\lim_{k\to\infty} f(k) = \infty$.

Lower bound (2)

- Theorem. (Huizanga, 2008) There is an unbounded function f such that $\chi_a(G) \geq f(\chi(G))$ for almost every graph G.
- Theorem. (Molloy and Thron, 2011) There is a function h tending to infinity such that

$$ch_a(G) \geq h(ch(G)).$$

- Theorem. (BZ 2013)

$$\chi_a(G) \geq K \log\log \chi(G)$$

where K is a positive integer.

Still more questions

- Problem. (Molloy, Thron) Are there any graph G such that $\chi_a(G)$ is less than the order of $\sqrt{\chi(G)}$?
- Problem. Can the lower bound $\log\log\chi(G)$ be improved?