Several Extensions of Keen Model

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Abstract

In this paper, we firstly review the derivation of Keen Model and show its trajectory. Then, we incorporated Keen Model with loan depreciation rate as well as a variable loan interest rate and we find a limit cycle. Finally, by using stock-flow consistency table, we further extended the model and examined the development of banks net worth, also a limit cycle comes out.

1 Review of Keen model

We start with a model for wage share and employment rate proposed by Goodwin. Besides full capital utilization, Goodwin assumes that all wages are consumed and all profits are invested. Then he presents the dynamics of wage share ω and employment λ as the following two-dimension differential equations:

$$\begin{cases} \dot{\omega} = \omega[\Phi(\lambda) - \alpha] \\ \dot{\lambda} = \lambda \left[\dfrac{1-\omega}{\nu} - \alpha - \beta - \delta \right] \end{cases} \tag{1}$$

where α and β is the growth rate of productivity and population respectively, ν is the capital–output ratio, δ is the capital deprecian rate and $\Phi(\lambda)$ is the Phillips Curve.

However, the assumption that all wages are consumed and all profits are reinvested doesn't necessarily hold in the real world. In fact, firms will borrow some money from the bank when economy is good, namely, firms have a high profit rate; and conversely, when economy is bad, they will only spend part or even none of their profit in reinvestment and rather pay off their debt. Thus, Keen introduces a banking sector to finance new investment.

Denoting by K the total capital stock, Y the total output and D the amount of debt, all in real term, the net profit after paying wages and interest on debt is

$$\Pi = (1 - \omega - rd) * Y \tag{2}$$

where r is a constant real interest rate and $d = D/Y$ is the debt ratio in economy. Then Keen formalizes the idea discussed above, which is actually the key insight provided by Minsky, by taking the change in capital stock to be

$$\dot{K} = \kappa(\pi) * Y - \delta K \tag{3}$$

where the rate of new investment is an increasing function of the new profit share $\pi = (1 - \omega - rd)$. Then he presents the following three-dimension ODE system which is known as Keen Model:

$$\begin{cases} \frac{\dot{\omega}}{\omega} = \Phi(\lambda) - \alpha \\ \frac{\dot{\lambda}}{\lambda} = \frac{\kappa(\pi)}{\nu} - \delta - \alpha - \beta \\ \frac{\dot{d}}{d} = \frac{\kappa(\pi) - \pi}{d} - \frac{\kappa(\pi)}{\nu} + \delta \end{cases} \tag{4}$$

Both mathematical analysis and simulation show that Keen Model has two locally stable non-trivial equilibriums: a good equilibrium with positive wage share, employment rate and finite debt ratio, which is an economic desirable way of development, and a bad equilibrium with zero wage share, zero eomplyment rate but infinite debt ratio, which is an undesirable economic state and every country want to avoid. We use simulation to illustrate such conclusion.

1.1 Examples

We choose the fundamental economic constants to be

$$\alpha = 0.015, \quad \beta = 0.01, \quad \delta = 0.06, \quad \nu = 3 \quad r = 0.03 \tag{5}$$

take the Phillips Curve to be

$$\Phi(\lambda) = \phi_0 + \frac{\phi_1}{(1-\lambda)^2} \tag{6}$$

with constants

$$\phi_0 = 0.005 \quad \phi_1 = 0.01 \times 0.07^2 \tag{7}$$

and take the invest propensity function as a sigmoid curve fomulated as

$$\kappa_5(\pi) = \frac{1}{1 + e^{-20(\pi - 0.256)}} \tag{8}$$

with its plot in Figure 1.

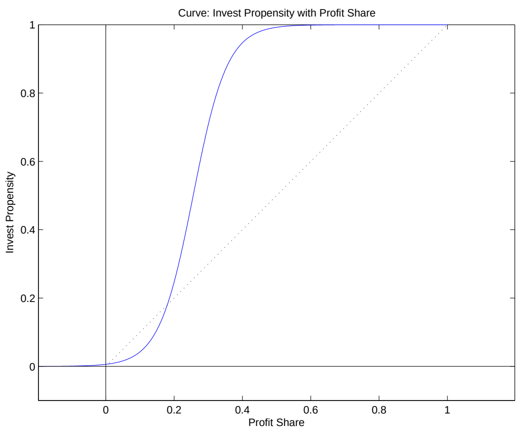


Figure 1: $\kappa(\pi)$

First, we let the initial condition to be

$$\omega_0 = 0.75, \quad \lambda = 0.92, \quad d = 0.5 \tag{9}$$

then the trajectory goes to the good equilibrium, which can be seen in Figure 2

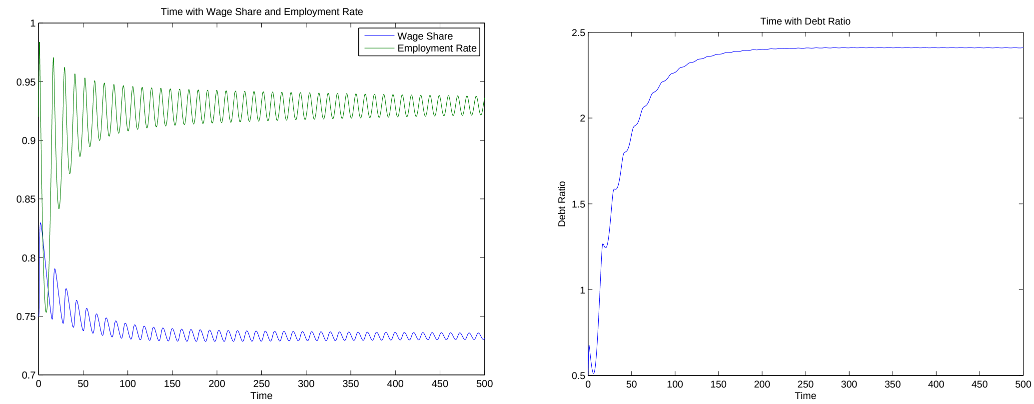


Figure 2: $\omega_0 = 0.75, \lambda_0 = 0.92, d_0 = 0.5$

However, if we take the initial condition to be

$$\omega_0 = 0.7, \quad \lambda = 0.9, \quad d = 0.5 \tag{10}$$

then the trajectory goes to the bad equilibrium, which can be seen in Figure 3

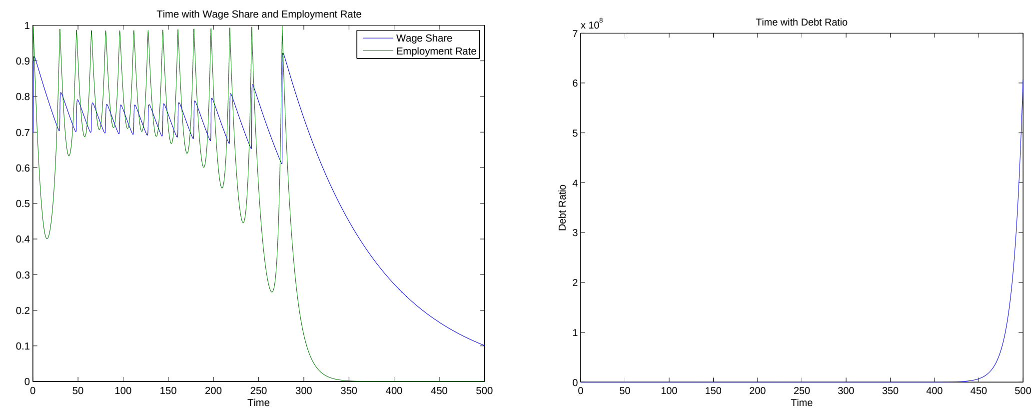


Figure 3: $\omega_0 = 0.7, \lambda_0 = 0.9, d_0 = 0.5$

2 Loan Default and Variable Interest Rate

As we have discussed, Keen Model has two equilibriums. However, we found it unrealistic about the bad equilibrium. This equilibrium shows that when economy is bad, debt will tends to infinity with no default. But what may happen is that the company would go bankcrupt and default their loans so that loan would not tend to infinity. Also we found it unrealistic about the assumption that there is a constant loan interest rate. So we incorporated Keen Model with depreciation rate of loan as well as a variable loan interest rate.

2.1 Derivation

We define δ_L as depreciation rate of loan(default), and we assume that

$$\delta_L = \delta_L(\pi, d) \tag{11}$$

is a decreasing function of π and an increasing function of d, which means low profit rates high debt ratios make the firms are more likely to default their loans; Moreover, we assume $\dot{r}$ is an increasing function of δ_L and a decreasing function of r, which means that when depreciation rate is at a high level relative to the instant loan interest rate, r will grow up. Now with the newly introduced δ_L, we have

$$\dot{D} = I - \Pi - \delta_L * D, \tag{12}$$

then

$$\frac{\dot{d}}{d} = \frac{\dot{D}}{D} - \frac{\dot{K}}{K} = \frac{\kappa(\pi) - \pi}{d} - \delta_L - \frac{\dot{K}}{K} \tag{13}$$

So we have a four-dimensional ODE system

$$\begin{cases} \dot{\omega} = \omega(\phi(\lambda) - \alpha) \\ \dot{\lambda} = \lambda(\frac{\dot{K}}{K} - \alpha - \beta) \\ \dot{d} = -d(\frac{\dot{K}}{K} + \delta_L) + \kappa(\pi) - \pi \\ \dot{r} = \dot{r}(\delta_L) \end{cases} \tag{14}$$

where $\frac{\dot{K}}{K} = \frac{\kappa(\pi)}{\nu} - \delta$.

Additionally, a suggested form of the function δ_L and $\dot{r}$ could be

$$\delta_L = (-0.0058 + \frac{0.00614}{\pi + 0.058}) * d \tag{15}$$

$$\dot{r} = A(1.5\delta_L - r) \tag{16}$$

with A as a coefficient that indicates the strength of the relationship. The plot of δ_L in relation to π when $d = 1$ can be seen in Figure 4.

2.2 Example

We use simulation to examine the stability of this system. From the simulation result of the original Keen Model, we can see that with the realistic initial value (wage share 70%, employment rate 90% and debt ratio 50%), the economy will slowly tend bad and the economic crisis come out. In constrast to the original Keen Model, we can see from the following simulation result that this extended model is more stable.

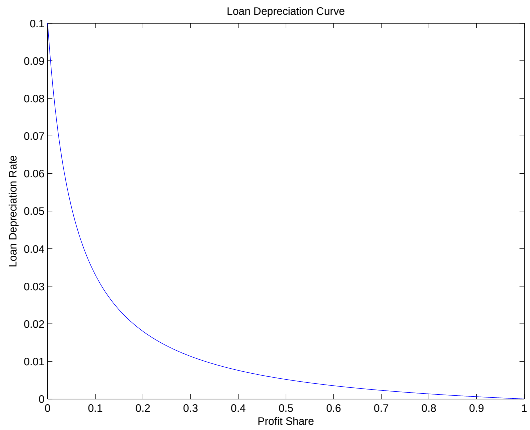


Figure 4: Loan Depreciation Ratio

For constrast propose, we adopt the same parameters and initial conditions as in the simulations of orinigal Keen Model. Also with the suggested forms of δ_L and $\dot{r}$ mentioned above and take $A = 1$, the simulation results with initial condition $(\omega_0, \lambda_0, d_0, r_0)$ as $(0.75, 0.92.0.5, 0.03)$ and $(0.7, 0.9.0.5, 0.03)$ are shown in Figure 5 and Figure 6 respectively, which we can find a limit cycle.

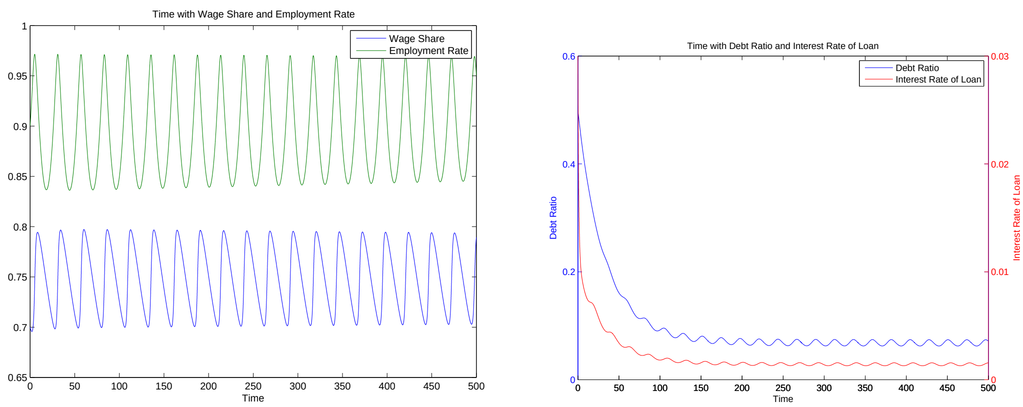
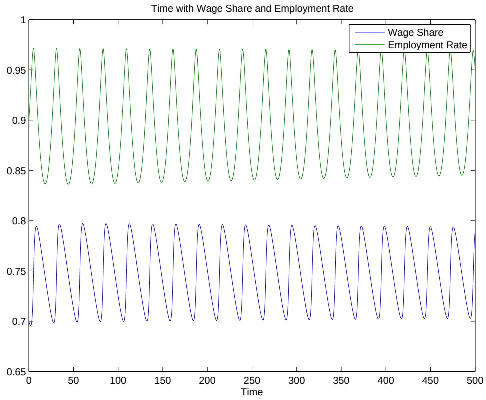


Figure 5: $\omega_0 = 0.75, \lambda_0 = 0.92, d_0 = 0.5, r_0 = 0.03$

3 Further Extension: Examine Banks' networth

In the former extension, we kind of consider bank and households as a whole, which means that banks lend all the money that households deposit to the firms and we haven't discussed whether bank would make some profits in the progress or whether banks have their own networth.

In this section, we split the bank sector from household and consider banks as companies that have their own networth and can make profit from the difference of loan interest rate and deposit interest rate.



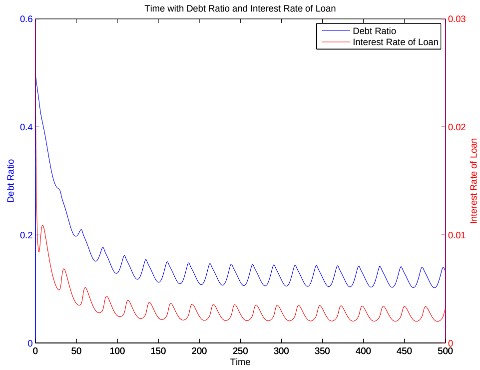


Figure 6: $\omega_0 = 0.7, \lambda_0 = 0.9, d_0 = 0.5, r_0 = 0.03$

3.1 Derivation

Now the relationship among household, firms and banks as well as their transactions can be seen from the following Stock-Flow Consistency Chart.

	Households	Firms(cur)	Firms(cap)	Bank(cur)	Bank(cap)	Total
Consumption	$-C$	$+C$				0
Investment		$+I$	$-I$			0
GDP		[Y]				
Wages	$+W$	$-W$				0
Interest on M	$+r_M.M_h$	$+r_M.M_f$		$-r_M.M$		0
Interest on L		$-r_L.L$		$+r_L.L$		0
Profit		$-\Pi$	$+\Pi$	$-\Pi_b$	$+\Pi_b$	0
Loan Deprecian			$+\delta_L.L$		$-\delta_L.L$	0
Financial Balance	$[S_h](=\dot{M}_h)$	[0]	$(-\dot{D})$	[0]	$[S_b](=\dot{K}_b)$	0
Deposits	$-\dot{M}_h$		$-\dot{M}_f$		$+\dot{M}$	0
Loans			$\dot{L}$		$-\dot{L}$	0
Totals	0	0	0	0	0	0

From the table above, we can easily get that

$$L = M + K_b = M_f + M_h + K_b \tag{17}$$

which means all the loans come from three part, deposit of firm, deposit of household and bank networth. Moreover, just like in Keen Model, we define firm debt $D := L - M_f = M_h + K_b$, debt-output ratio $d := \frac{D}{Y}$ household deposit ratio $d_h := \frac{M_h}{Y}$ and bank networth ratio $d_b := \frac{K_b}{Y}$

To simplify the model, we only consider the case in which firms don't deposit money in the banks, namely $M_f \equiv 0$, then we have $D = L = M_h + M_b (\Leftrightarrow d = d_h + d_b)$. Based on this and the same as the formor extended Keen Model, we get

$$\frac{\dot{d}}{d} = \frac{\kappa(\pi) - \pi}{d} - \frac{\dot{K}}{K} - \delta_L \tag{18}$$

with $\pi = 1 - \omega - d \cdot r_L$ and $\dot{K} = I - \delta K = \kappa(\pi) \cdot Y - \delta K$

Also, we assume that

$$r_M \cdot M_h = \frac{2}{3} \cdot (r_L - \delta_L) \cdot D \tag{19}$$

and since

$$\dot{M}_h = W + r_M \cdot M_h - C \tag{20}$$

then

$$\frac{\dot{M}_h}{M_h} = \frac{W - C}{M_h} + r_M \tag{21}$$

so

$$\frac{\dot{d}_h}{d_h} = \frac{\dot{M}_h}{M_h} - \frac{\dot{Y}}{Y} = \frac{\omega + \kappa(\pi) - 1}{d_h} + r_M - \frac{\dot{K}}{K} \tag{22}$$

Thus we get a five-dimesional ODE system

$$\begin{cases} \dot{\omega} = \omega(\phi(\lambda) - \alpha) \\ \dot{\lambda} = \lambda(\frac{\dot{K}}{K} - \alpha - \beta) \\ \dot{d} = -d(\frac{\dot{K}}{K} + \delta_L) + \kappa(\pi) - \pi \\ \dot{r_L} = \dot{r_L}(\delta_L) \\ \dot{d_h} = d_h(r_M - \frac{\dot{K}}{K}) + \kappa(\pi) - 1 + \omega \end{cases} \tag{23}$$

with $\frac{\dot{K}}{K} = \frac{\kappa(\pi)}{\nu} - \delta$, $\delta_L = \delta_L(\pi, d)$, and $r_M = \frac{2}{3} \cdot \frac{d}{d_h} \cdot (r_L - \delta_L)$

Note: we can also get

$$\dot{d_b} = \dot{d} - \dot{d_h} = d_h(r_L - r_M - \delta_L) + d_b(r_L - \delta_L - \frac{\dot{K}}{K}) \tag{24}$$

from the Stock-Flow Consistency Chart and substitute the function about $\dot{d_h}$ with this one in Equation (23).

3.2 Example

Now with the newly extended model about banks, we can evaluate the bank net worth in the preceding examples.

Again with the same parameter values and let the initial condition$(\omega_0, \lambda_0, d_0, r_0)$ to be $(0.75, 0.92.0.5, 0.03)$ or $(0.7, 0.9.0.5, 0.03)$, we cam see the fluctuation of bank net worth ratio and household deposit ratio in contrast with total debt ratio and loan interest rate and deposit interest rate in Figure (7) and Figure (8).

From both cases, we can see that we economy goes well, the bank net worth come to a low level relative with household deposit and both loan and deposit interest rate would gradually diminish.

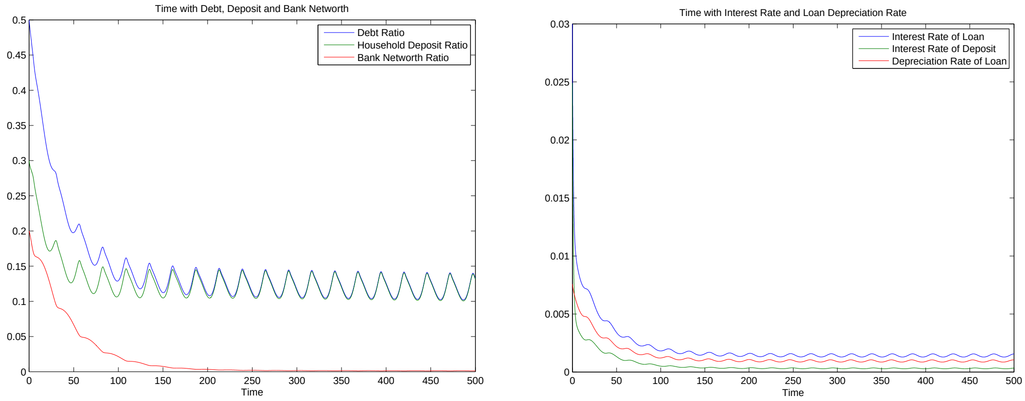


Figure 7: $\omega_0 = 0.75, \lambda_0 = 0.92, d_0 = 0.5, r_0 = 0.03$

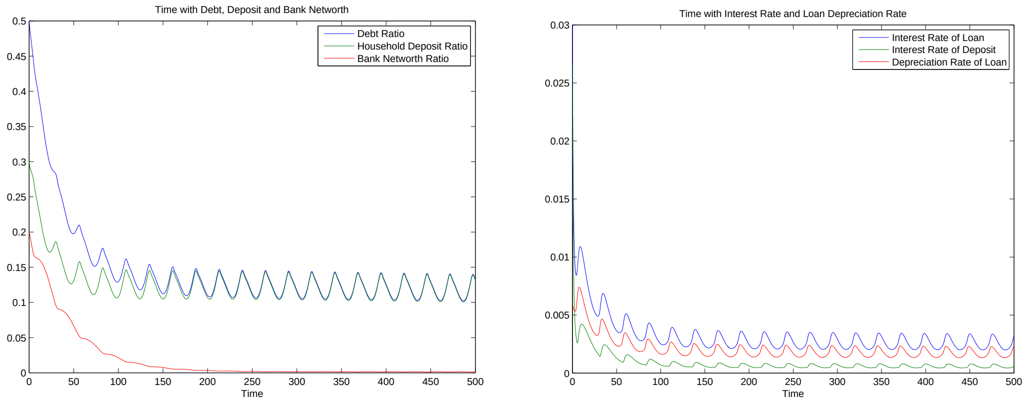


Figure 8: $\omega_0 = 0.7, \lambda_0 = 0.9, d_0 = 0.5, r_0 = 0.03$

4 References

1. M. R. Grasselli, An analysis of the Keen model for credit expansion, asset price bubbles and financial fragility, Math Finan Econ, DOI 10.1007/s11579-012-0071-8