Greg Hjorth, 1963–2011.

Elliott's program and descriptive set theory II

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(joint work with Andrew Toms and Asger Törnquist and with Sam Coskey, George Elliott and Martino Lupini)

York University

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A detailed presentation of the material from talk #1 and most of talk #2 is available in my lecture notes from the Singapore Summer School, available upon request.

The plan

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    1.1 Basic properties of C*-algebras.
    1.2 Classification: UHF and AF algebras.
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3. Saturday: Convincing you that 1.2–1.3 is logic.

Basic definitions

A topological space X is *Polish* if it is separable and completely metrizable.
A subset of X is *analytic* if it is a continuous image of a Borel set.
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The only classical-ish example for which this does not seem to work is homeomorphism relation of Polish spaces.

Smoothness

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Similarity of $n \times n$ complex Hermitian matrices is smooth. Associate to M the list of its eigenvalues (in the increasing order, with multiplicities).

A criterion for non-smoothness

Proposition

If $G \curvearrowright X$ is a Polish group action on a Polish space such that all orbits are dense and meager (i.e., of first category) then the orbit equivalence relation E_G^X is not smooth.

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Proof.

If $f\colon X \to \mathbb{R}$ is Borel, then we can find a dense G_δ subset Y of X such that the restriction of f to Y is continuous. The set $\{x \in X : g.x \in Y\}$ is comeager for all $g \in G$. Therefore we can find $x \in X$ such that $\{g \in G : g.x \in Y\}$ is comeager in G. Therefore $[x] \cap Y$ is dense. Then f is constant on $[x]$ and (by continuity) on Y. $\square$

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Example (Vitali equivalence relation)

On $\mathbb{R}$ let $x \sim y$ iff $x - y \in \mathbb{Q}$. All orbits are countable and dense, hence $\sim$ is not smooth.

Borel reducibiity

Definition (H. Friedman, Kechris)

Assume E, F are equivalence relations on Polish spaces X, Y, respectively. Then E is *Borel reducible* to F, or $E \leq_B F$, if there is a Borel-measurable map $f\colon X \to Y$ such that

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Interpretations:

1. Borel cardinality of X/E is $\leq$ than the Borel cardinality of Y/F.
2. Classification problem for E is simpler than the classification problem for F.
3. F-Equivalence classes are complete invariants for E-equivalence classes.

Glimm-Effros Dichotomy

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This is false for analytic equivalence relations, since there is one with exactly $\aleph_1$ equivalence classes.
Combinatorics of the proof comes from Glimm's theorem to the effect that every non-type I C*-algebra has M_{2^∞} as a subquotient.

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In almost all cases, the space of invariants has a Polish topology and the computation of invariants is given by a Borel-measurable function.

Example 1: Polish space of countable groups

Every countable group G is isomorphic to one of the form $(\mathbb{N}, \cdot_G)$, and the latter is coded by

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Therefore the space $\mathbb{G}$ of countable discrete groups is a Borel subspace of the compact metric space $\mathcal{P}(\mathbb{N}^3)$.

The isomorphism $\cong^G$ is an analytic equivalence relation, because (by S_∞ we denote the Polish group of all permutations of $\mathbb{N}$)

$$\{(G, H, f) \in \mathbb{G}^2 \times S_\infty, f\colon G \to H \text{ is an isomorphism}\}$$

is Borel.

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Models of a fixed first-order theory form a Borel set.
The isomorphism relation is an S_∞-orbit equivalence relation.

Classification by countable structures

An equivalence relation (X, E) is *classified by countable structures* if there is a countable language L and a Borel map f from X into the space of countable L-models such that

$$x \mathrel{E} y \text{ iff } f(x) \cong f(y).$$

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But what if we are classifying structures that are merely separable instead of countable?

Hyperspaces

Assume (K, d) is a compact metric space. The space $F(K)$ of all compact subsets of K equipped with the Hausdorff metric

$$d(F, G) = \inf\{\varepsilon : F \subseteq_\varepsilon G \text{ and } G \subseteq_\varepsilon F\}$$

(with $F \subseteq_\varepsilon G$ iff $(\forall a \in F)(\exists b \in G) d(a, b) \leq \varepsilon$) is also compact.

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This does not work for non-compact Polish spaces X, since the Hausdorff metric is not separable on $F(X)$.

Effros Borel space

For a Polish space X let $F(X)$ be the space of closed subsets of X. Consider σ-algebra Σ on $F(X)$ generated by sets

$$\{A \in F(X) | A \cap U \neq \emptyset\}$$

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Theorem (Effros)

$(F(X), \Sigma)$ is a standard Borel space (i.e., Σ is the σ-algebra of Borel sets for some Polish topology on X).

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Example

Every separable Banach space is isometric to a closed subspace of $C([0,1])$. Therefore

$$\{X \in F(C([0,1])) : X \text{ is a closed subspace}\}$$

is 'the standard Borel space of all separable Banach spaces.'

Urysohn space, $\mathbb{U}$

This is a separable complete metric space which is universal for separable metric spaces and satisfies the following extension property:
for all finite metric $X \subseteq Y$, every isometry $f\colon X \to \mathbb{U}$ extends to an isometry $g\colon Y \to \mathbb{U}$.

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{U} \\ Y \arrow[u, "\subseteq"] \arrow[ur, "g"'] & \end{tikzcd}$$

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Theorem (Clemens, Gao–Kechris, 2000)

Translation action of the isometry group $\mathrm{Iso}(\mathbb{U})$ *on* $F(\mathbb{U})$ *is the maximal orbit equivalence relation of a Polish group action.*

C*-algebras review

1. Concrete C*-algebra is a norm-closed algebra of operators on a complex Hilbert space.
2. (Gelfand–Naimark–Segal, GNS) Abstract C*-algebra is a Banach algebra with involution * that satisfies $\|a\|^2 = \|aa^*\|$ for all a.
3. (Gelfand–Naimark) Compact metric spaces are complete isomorphism invariants for separable unital abelian C*-algebras.

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7. (Elliott, 1974) Pre-ordered abelian group $\mathbf{K}_0$ is a complete isomorphism invariant for AF algebras.
8. In (3), (5), and (7) we even have equivalence of categories.

A nonseparable digression

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(This is not going to be on the exam.)

Standard Borel space of C*-algebras, I

Lemma

Every separable C-algebra is isomorphic to a subalgebra of $\mathcal{B}(H)$, for* **the** *separable, infinite-dimensional complex Hilbert space H.*

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Theorem (Junge–Pisier, 1995)

There is no universal separable C-algebra.*

Standard Borel space of C*-algebras, II

Definition (Kechris, 1995)

Endow $\mathcal{B}(H)$ with the Borel structure of the strong operator topology. Then $\Gamma = \mathcal{B}(H)^{\mathbb{N}}$ is a standard Borel space.
Every $\gamma \in \Gamma$ 'codes' the C*-algebra $C^*(\gamma)$ generated by it.

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Proposition

The following subsets of Γ are Borel.

1. *(easy)* $\{\gamma | C^*(\gamma) \text{ is unital}\}$,
2. *(Effros)* $\{\gamma | C^*(\gamma) \text{ is nuclear}\}$,
3. *(F.–Toms–Törnquist)* $\{\gamma | C^*(\gamma) \text{ is simple}\}$.

Review: Elliott's program

Conjecture (Elliott, 1990's)

All nuclear,[1] *separable, simple, unital, infinite-dimensional C*-algebras are classified by the K-theoretic invariant,*

$$\mathrm{Ell}(A): \qquad ((K_0(A), K_0(A)^+, 1), K_1(A), T(A), \rho_A).$$

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The conjecture is false, but it has led to some spectacular mathematics and many instances od its revised version have been confirmed.

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Everything is Borel

Theorem (F.–Toms–Törnquist, 2011)

There is a standard Borel spaces **Ell** *of Elliott invariants, and the computation of* Ell *is given by a Borel map.*

$$\text{Separable C*-algebras} \twoheadleftarrow \Gamma \xrightarrow{\Phi} \mathbf{Ell} \twoheadrightarrow \text{Elliott invariants}$$

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Corollary

The isomorphism relation of unital UHF algebras is smooth.
The isomorphism relation of AF algebras is classifiable by countable structures.

Proof.

Combine the above with Glimm's and Elliott's theorems. $\square$

Hjorth's turbulence

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3. $\forall x \in X$, $\forall U \in \mathcal{O}(e)$, $\forall V \in \mathcal{O}(x)$ the graph on U defined by

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   is such that the closure of the connected component of x intersects every orbit.

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Example

The action of $c_0 = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \lim_n |x_n| = 0\}$ on $\mathbb{R}^{\mathbb{N}}$ by translation.

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Theorem (Hjorth, 1997)

If $G \curvearrowright X$ is turbulent then the orbit equivalence relation E_G^X is not classified by countable structures.

Compact metrizable spaces I

Proposition (Folklore?)

Homeomorphism relation of closed subsets of $[0,1]$ *is classifiable by countable structures.*

Proof.

If $K \subseteq [0,1]$ is compact then it has only two types of connected components: singleton and interval. Use the 'tagged' version of Cantor–Bendixson analysis. $\square$

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Corollary (trust me)

The isomorphism relation of unital abelian C-algebras generated by a single self-adjoint element is classifiable by countable structures.*

Compact metrizable spaces II

Proposition (F.–Toms–Törnquist, after Hjorth)

Homeomorphism relation of closed subsets of $[0,1]^2$ *is not classifiable by countable structures.*

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Corollary

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Question

Does the complexity of the isomorphism relation for unital abelian separable C-algebras increase if the number of generators increases?*

AI algebras are not classifiable by countable structures

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Theorem (F.–Toms–Törnquist, 2011)

We have the following Borel-reductions

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\begin{tikzcd}
\text{graph isomorphism} \arrow[r] & \text{compact metric spaces} \arrow[l, bend right, "/"'] \arrow[r] \arrow[d, leftrightarrow] & \text{simple unital AI algebras} \arrow[l, dotted, bend right, "?"'] \arrow[r] & \text{Elliott's invariants} \arrow[l, dotted, bend right, "?"'] \\
\text{countable structures} \arrow[u] & \text{unital abelian C*-algebras} & &
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AI algebras are not classifiable by countable structures ...although they are classifiable by Elliott's invariant

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The dark side

On $[0,1]^{\mathbb{N}}$ define

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Theorem (Ferenczi–Louveau–Rosendal, 2009)

Isomorphism of separable Banach spaces is the $\leq_B$-maximal analytic equivalence relation.

Together with an another result of F.–Toms–Törnquist, this gives

$$E_0 \longrightarrow E_1 \longrightarrow E_{K_\sigma} \longrightarrow \begin{array}{l}\text{biembeddability}\\ \text{of AF}\\ \text{algebras}\end{array} \longrightarrow \begin{array}{l}\text{isomorphism}\\ \text{of Banach}\\ \text{spaces}\end{array}$$

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Theorem (folklore)

Isomorphism of von Neumann factors with separable predual is below an orbit equivalence relation.

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Therefore classifying von Neumann factors is easier than classifying Banach spaces (up to the isomorphism).[2]

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Theorem (F.–Toms–Törnquist, 2011)

The isomorphism relations in the following categories are below an orbit equivalence relation.

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Proof of (1) uses a Borel version of a very difficult result of Kirchberg and does not appear to be amenable to generalizations.

Polish groupoids

Partially following A. Ramsay, we say that a structure $(\mathcal{O}, \mathcal{A})$ (objects and arrows) is a *Polish groupoid* if

1. It is a groupoid,
2. Both $\mathcal{O}$ and $\mathcal{A}$ carry a Polish topology,
3. Operations $s \colon \mathcal{A} \to \mathcal{O}$ and $r \colon \mathcal{A} \to \mathcal{O}$ ('source' and 'range') are continuous,
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Proposition (Coskey–Elliott–F.–Lupini, 2012)

$E_1 \not\leq_B E_{(\mathcal{O},\mathcal{A})}$ for any Polish groupoid.

However. . .

Question

If A is a separable C-algebra, does the groupoid whose objects are subalgebras of A and arrows are *-isomorphisms between them carry a Polish groupoid structure?*

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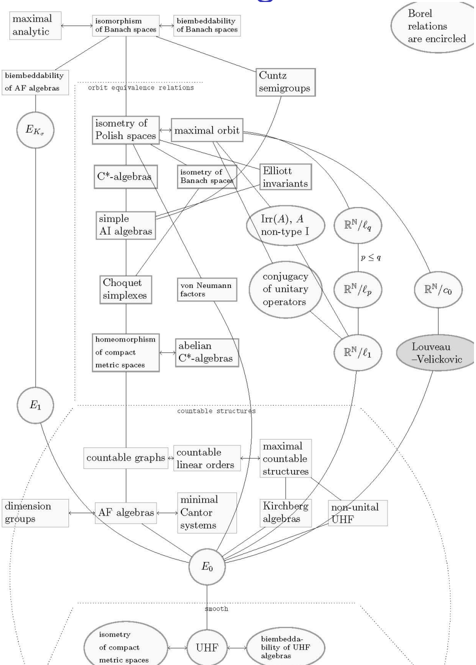
We will find this out tomorrow.

Borel reductions diagram



Borel reductions diagram, sideways