Convex Sets Associated to C*-Algebras

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Classical Situation (1970's): $\text{Ext}(\mathfrak{A})$

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We use isomorphisms between $R^{\mathcal{U}}$ and $pR^{\mathcal{U}}p$ for p a projection in $R^{\mathcal{U}}$ to define convex combinations.

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$$t[\pi] + (1-t)[\rho] = \left[\left(\begin{array}{c|c} p\pi p & 0 \\ \hline 0 & p^{\perp}\rho p^{\perp} \end{array} \right) \right]$$

where p is a projection in $R^{\mathcal{U}}$ and $\tau_R(p) = t$.

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With this definition, we may consider $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ as a closed, bounded, convex subset of a Banach space.

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Theorem (Brown, 2011)

$[\pi] \in \mathbb{H}om(N, R^{\mathcal{U}})$ is extreme if and only if $\pi(N)' \cap R^{\mathcal{U}}$ is a factor.

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Preliminaries

Definition

For a separable, unital, tracial C*-algebra $\mathfrak{A}$, and a separable McDuff II$_1$-factor M ($M \cong M \otimes R$), we define $\mathbb{H}\text{om}(\mathfrak{A}, M)$ to be the space of unital $*$-homomorphisms $\pi : \mathfrak{A} \to M$ modulo the equivalence relation of weak approximate unitary equivalence (w.a.u.e.).

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That is, $[\pi] = [\rho]$ if there is a sequence $\{u_n\}$ of unitaries in M such that for every $a \in \mathfrak{A}$ we have

$$\lim_n ||\pi(a) - u_n \rho(a) u_n^*||_2 = 0.$$

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We endow $\mathbb{H}\mathrm{om}(\mathfrak{A}, M)$ with the topology of pointwise convergence (with appropriate consideration for equivalence classes).

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For a McDuff factor M, an isomorphism $\sigma_M : M \otimes R \to M$ is a *regular isomorphism* if $\sigma_M \circ (\mathrm{id}_M \otimes 1_R) \sim \mathrm{id}_M$.

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Definition

For $t \in [0,1], [\pi], [\rho] \in \mathbb{H}\mathrm{om}(\mathfrak{A}, M)$, we define

$$t[\pi] + (1-t)[\rho] := [\sigma_M(\pi \otimes p + \rho \otimes p^{\perp})]$$

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(Correct) Picture:

$$\left(\begin{array}{c|c} (1_M \otimes p)(\pi \otimes 1_R)(1_M \otimes p) & 0 \\ \hline 0 & (1_M \otimes p^{\perp})(\rho \otimes 1_R)(1_M \otimes p^{\perp}) \end{array}\right)$$

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Given $\pi : \mathfrak{A} \to M$, we have that $\mathbb{H}om(\mathfrak{A}/ker\pi, M)$ is a face of $\mathbb{H}om(\mathfrak{A}, M)$.

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Connection to Trace Space

Given $[\pi] \in \mathbb{H}\text{om}(\mathfrak{A}, M)$ we get a (unital) trace on $\mathfrak{A}$ given by

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The correspondence $[\pi] \mapsto \tau_M \circ \pi$ is well-defined, continuous, and affine.

Natural question: For a fixed M, does this give all of the (unital) traces on $\mathfrak{A}$?

Nuclear Case

Theorem (A.)

If $\mathfrak{A}$ is nuclear then for any McDuff M we have $\mathbb{H}om(\mathfrak{A}, M) \cong T(\mathfrak{A})$ given by $[\pi] \leftrightarrow \tau_M \circ \pi$.

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English Version: All traces of a separable unital nuclear algebra "lift" through any fixed McDuff factor; and the traces "remember" their homomorphisms up to w.a.u.e.

(Recall: $\mathfrak{A}$ nuclear $\Rightarrow$ Ext($\mathfrak{A}$) is a group. But the class of algebras $\mathfrak{A}$ for which Ext($\mathfrak{A}$) is a group is strictly larger than the nuclears. In 1977 Anderson showed that Ext($\mathfrak{A}$) is not always a group.)

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1. Well-Behaved Non-Nuclear: A non-nuclear $\mathfrak{A}$ where for any McDuff M, all traces lift through M and the traces remember their homomorphisms.
2. Too Many Traces: A necessarily non-nuclear algebra $\mathfrak{B}$ where for some McDuff M, there is a trace on $\mathfrak{B}$ that does not lift through M.

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Similar to the program of $\mathrm{Ext}(\mathfrak{A})$ we would like to find examples of the following.

1. Well-Behaved Non-Nuclear: A non-nuclear $\mathfrak{A}$ where for any McDuff M, all traces lift through M and the traces remember their homomorphisms.
2. Too Many Traces: A necessarily non-nuclear algebra $\mathfrak{B}$ where for some McDuff M, there is a trace on $\mathfrak{B}$ that does not lift through M.
3. Forgetful Trace: A necessarily non-nuclear algebra $\mathfrak{C}$ where for some McDuff M, there is a trace on $\mathfrak{C}$ lifting through M via two inequivalent homomorphisms.

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Also, it can be shown that the traces remember their homomorphisms (in any McDuff).

(2) Too Many Traces

Brown found an algebra $\mathfrak{B}$ such that for any McDuff M, there is a trace $T_M \in T(\mathfrak{B})$ so that the von Neumann closure of the GNS representation π_{T_M} is isomorphic to M. That is,

$$\pi_{T_M}(\mathfrak{B})'' \cong M.$$

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A result by Ozawa demonstrates that there is no separable universal McDuff factor. So there cannot be an M such that *every* trace of $\mathfrak{B}$ lifts through M.

(3) Forgetful Trace

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A result of Hadwin's using the concept of dimension ratio (related to free entropy) gives a II_1 factor N and two inequivalent homomorphisms $\pi, \rho : C_r^*(\mathbb{F}_2) \to N$ such that $\tau_N \circ \pi = \tau_N \circ \rho$.

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Alas, N is not necessarily McDuff.

Question: Is the inequivalence of π and ρ preserved when we pass to $\pi \otimes 1_R, \rho \otimes 1_R : C_r^*(\mathbb{F}_2) \to N \otimes R$?