Noncommutative Uncertainty Principle

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Classical Uncertainty Principles

- Heisenberg uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

- Hirschman uncertainty principle:

$$H_s(|f|^2) + H_s(|\hat{f}|^2) \geq 0.$$

- Donoho-Stark uncertainty principle:

$$|\mathrm{supp}(f)||\mathrm{supp}(\hat{f})| \geq |G|.$$

Heisenberg Uncertainty Principle

- Heisenberg [1927]

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

x position; p momentum.

- A mathematic formulation:

$$(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx)(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi) \geq \frac{\|f\|_2^4}{16\pi^2},$$

where $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$

Hirschman-Beckner Uncertainty Principle

- Hirschman [1957], for real number group $\mathbb{R}$ and $\|f\|_2 = 1$

$$H_s(|f|^2) + H_s(|\hat{f}|^2) \geq 0. \tag{1}$$

Shannon Entropy: $H_s(|f^2|) = -\int_{-\infty}^{\infty} |f|^2 \log |f|^2 dx.$

- Hirschman's Conjecture:

$$H_s(|f|^2) + H_s(|\hat{f}|^2) \geq \log \frac{e}{2}.$$

- Beckner [1975], the conjecture is true.
- Özaydin and Przebinda [2000],

locally compact abelian group with an open compact subgroup for inequality (1).

Donoho-Stark Uncertainty Principle

• Donoho-Stark [1989], for cyclic group G,

$$|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G|.$$

f, a function on G
$\hat{f}$, the Fourier transform of f
$|\text{supp}(f)| = \#\{x \in G | f(x) \neq 0\}$.
• K. Smith [1990], finite abelian group
• Özaydin and Przebinda [2000],
locally compact abelian group with an open compact subgroup

Some recent results:
[D. Goldstein, R. Guralnick, and I. Isaacs, 2005] finite groups
[J. Crann and M. Kalantar, 2014] Kac algebras (C^* Hopf algebras or quantum groups in von Neumann algebraic setting)
Subfactor theory naturally provides a Fourier transform over a pair of von-Neumann algebras and a measurement.
W. Szymanski [1994], irreducible depth-2 subfactors $\leftrightarrow$ Kac algebras.
We are going to talk about the uncertainty principle for finite index subfactors.

- Haussdorff-Young's inequality
- Young's inequality (new for Kac algebras)
- Uncertainty principles
- Minimizers (new for finite non-abelian groups)

Subfactors

Theorem (Jones83)

$$\{[\mathcal{M}:\mathcal{N}] := \dim_{\mathcal{N}}(L^2(\mathcal{M}))\} = \{4\cos^2\frac{\pi}{n}, n=3,4,\cdots\} \cup [4,\infty].$$

- $\mathcal{N} \subset \mathcal{M}$, a subfactor (of type II_1) with finite index
- Jones' projection $e_1 \in B(L^2(\mathcal{M})) : L^2(\mathcal{M}) \to L^2(\mathcal{N})$
- Basic construction $\mathcal{M}_1 = <\mathcal{M}, e_1>''$
- Jones tower

$$\mathcal{N} \subset \mathcal{M} \overset{e_1}{\subset} \mathcal{M}_1 \overset{e_2}{\subset} \mathcal{M}_2 \overset{e_3}{\subset} \cdots$$

- Standard invariant

$$\begin{array}{ccccccccc}
\mathcal{N}'\cap\mathcal{N} & \subset & \mathcal{N}'\cap\mathcal{M} & \subset & \mathcal{N}'\cap\mathcal{M}_1 & \subset & \mathcal{N}'\cap\mathcal{M}_2 & \subset & \cdots \\
 & & \cup & & \cup & & \cup & & \\
 & & \mathcal{M}'\cap\mathcal{M} & \subset & \mathcal{M}'\cap\mathcal{M}_1 & \subset & \mathcal{M}'\cap\mathcal{M}_2 & \subset & \cdots
\end{array}$$

Axioms of standard invariants

- Ocneanu's paragroups [1988]
- Popa's standard λ-lattices [1995]
- Jones' subfactor planar algebras [1999]

Group Subfactors

Example

When $\mathcal{M} = \mathcal{N} \rtimes G$, for an outer action a finite abelian group G, we have $[\mathcal{M} : \mathcal{N}] = |G|$,
the Jones tower

$$\mathcal{N} \subset \mathcal{N} \rtimes G \subset \mathcal{N} \rtimes G \rtimes \hat{G} \subset \mathcal{N} \rtimes G \rtimes \hat{G} \rtimes G \subset \cdots$$

and the standard invariant

$$\begin{array}{ccccccccc} \mathbb{C} & \subset & \mathbb{C} & \subset & L\hat{G} & \subset & L(\hat{G} \rtimes G) & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} & \subset & \mathbb{C} & \subset & LG & \subset & \cdots \end{array}$$

The 2-box spaces of the standard invariant ($\mathcal{M}' \cap \mathcal{M}_2$ and $\mathcal{N}' \cap \mathcal{M}_1$) recover the group G and its dual $\hat{G}$!
Moreover $\mathcal{N}' \cap \mathcal{M}_2$ provides a natural algebra to consider G and $\hat{G}$ simultaneously!

A Pair of C^* Algebras $(\mathcal{N}' \cap M_1, \mathcal{M}' \cap M_2)$

- Measure: the (unnormalized) trace of $\mathcal{M}_n$
- p-norm: $\|x\|_p = tr(|x|^p)^{\frac{1}{p}}, p \geq 1$
- Fourier transform (Ocneanu): 1-click rotation (for paragroups)

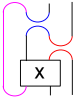
Definition

Fourier transform $\mathcal{F} : \mathcal{N}' \cap M_1 \to \mathcal{M}' \cap M_2$,

$$\mathcal{F}(x) = [\mathcal{M} : \mathcal{N}]^{\frac{3}{2}} E_{\mathcal{M}'}^{N'}(x e_2 e_1)$$

for $x \in \mathcal{N}' \cap M_1$, where $E_{M'}^{N'}$ is the trace preserving condition expectation from $\mathcal{N}'$ to $\mathcal{M}'$.

In subfactor planar algebras, the fourier transform is a 1-click rotation,



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Main Theorem (Haussdorf-Young's inequality)

For an irreducible subfactor $\mathcal{N} \subset \mathcal{M}$ with finite index, take $\delta = \sqrt{[\mathcal{M}:\mathcal{N}]}$.
For any $x, y \in \mathcal{N}' \cap \mathcal{M}_1$, we have

Theorem (Jiang-L-Wu)

$$\|\mathcal{F}(x)\|_p \leq \left(\frac{1}{\delta}\right)^{1-\frac{2}{p}} \|x\|_q \tag{2}$$

where $2 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

• Extremal: x is called extremal, if the equality of (2) holds.
All positive operators are extremal.
Our proof of Haussdorf-Young's inequality also works for Popa's λ-lattice, modular tensor category etc.

Main Theorem (Young's inequality)

- Convolution: $x * y = \mathcal{F}(\mathcal{F}^{-1}(x)\mathcal{F}^{-1}(y))$

Theorem (Jiang-L-Wu)

$$\|x * y\|_r \leq \frac{\|x\|_p \|y\|_q}{\delta}.$$

where $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

Main Theorem (Hirschman-Beckner uncertainty principle)

Theorem (Jiang-L-Wu)

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq \|x\|_2(2\log\delta - 4\log\|x\|_2),$$

where $H(|x|^2) = -tr_2(|x|^2 \log |x|^2)$ *is the von Neumann entropy of* $|x|^2$*.*

A quick proof.

Take the derivative of Hausdorff-Young's inequality at $p = 2$. □

Main Theorem (Donoho-Stark uncertainty principle)

Theorem (Jiang-L-Wu)

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq \delta^2,$$

where $\mathcal{S}(x)$ is the trace of range projection of x, $x \neq 0$.

A quick proof.

Take log on both side and apply the former Theorem at $\|x\|_2 = \delta^{\frac{1}{2}}$. The inequality reduces to the fact that $\log \mathcal{S}(x) \geq \log \mathcal{H}(|x|^2)$ which follows from the concavity of $-t \log t$. □

Minimizer of classical Uncertainty Principles

Suppose G is a finite abelian group, and $H < G$.

- Translation: $f(x) \mapsto f(x+y)$
- Modulation: $f(x) \mapsto \chi(x)f(x)$, where χ is a character of G
- Indicator function of H: $\sum_{h \in H} h$.

Theorem (Özaydin and Przebinda)

The follows are equivalent

(1) $H(|x|^2) + H(|\mathcal{F}(x)|^2) = \|x\|_2(2\log\delta - 4\log\|x\|_2)$

(2) $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = \delta^2$

(3) $x = c\sum_{h\in H}\chi(h)hg,\ c \neq 0,\ H < G,\ g \in G,\ \chi \in \hat{G}$

Remark

The generalization of these concepts is not obvious in the non-commutative world. That makes extra difficulties to characterize minimizers of uncertainty principles for subfactors.

Main Theorem (minimizers)

A nonzero element x is called an extremal bi-partial isometry if x and $\mathcal{F}(x)$ are multiples of extremal partial isometries.
A projection p is called a biprojection if $\mathcal{F}(p)$ is a multiple of a projection.
Biprojections are introduced by Bisch [1994] and studies by Bisch and Jones [1997] from planar algebra perspective. Biprojections generalize indicator functions of subgroups.
We introduced a new notion, a bi-shift of a biprojection, which generalizes a translation and a modulation of the indicator function of a subgroup.

Theorem (Jiang-L-Wu)

The following statements are equivalent,

(1) $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = \delta^2$;

(2) $H(|x|^2) + H(|\mathcal{F}(x)|^2) = \|x\|_2(2\log\delta - 4\log\|x\|_2)$;

(3) *x is an extremal bi-partial isometry;*

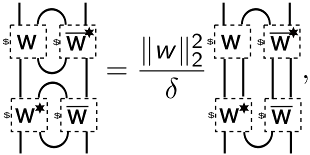
(3') *x is a partial isometry and $\mathcal{F}^{-1}(x)$ is extremal;*

(4) *x is a bi-shift of a biprojection.*

Remarks on Minimizers

To prove the theorem, we find the following key relation of a norm-1 extremal bi-partial isometry w (in planar algebras) based on Young's inequality :

$$(w^* * \overline{w})(w * \overline{w}^*) = \frac{\|w\|_2^2}{\delta}(w^* w) * (\overline{w}\ \overline{w}^*)$$

i.e. 

where $\overline{w} = \mathcal{F}^2(w)$.
Moreover $((w^*) * \overline{w})(w * \overline{w}^*)$ is a biprojection.
The relation is obtained by planar algebra methods. Up to now, we cannot find any other method to prove the above relation.

Main results (uniqueness)

Donoho and Stark 1989 noticed that the minimizer of uncertainty principles is uniquely determined by the supports of itself and its Fourier transform. This kind of result is very useful for signal recovery. It is further developed by Candes, Romberg and Tao 2006.

We are considering non-commutative algebras. Both an element and its Fourier transform have two supports.

Theorem (Jiang-L-Wu)

The minimizer of uncertainty principles is uniquely determined by the range projections of itself and its Fourier transform.

Minimizers of Groups

Proposition (Jiang-L-Wu)

Suppose G is a finite (non-abelian) group. Take a subgroup H, a one dimension representation χ of H, an element $g \in G$, a non-zero constant $c \in \mathbb{C}$. Then

$$x = c \sum_{h \in H} \chi(h) hg$$

is a bi-shift of a biprojection. Conversely any bi-shift of a biprojection is of this form.

Remark

Note that χ is the pull back of a character of $H/[H, H]$, where $[H, H]$ is the commutator subgroup.

Main results (for the pair $(\mathcal{N}' \cap M_{k-1}, \mathcal{M} \cap \mathcal{M}_k)$)

Definition (Ocneanu)

For any $x \in \mathcal{N}' \cap M_{k-1}$, the Fourier transform $\mathcal{F} : \mathcal{N}' \cap M_{k-1} \to \mathcal{M} \cap \mathcal{M}_k$ is

$$\mathcal{F}(x) = [M : N]^{\frac{n+1}{2}} E_{M'}^{N'}(x e_n e_{n-1} \cdots e_1).$$

Theorem (Jiang-L-Wu)

$$\|\mathcal{F}(x)\|_p \leq \left(\frac{1}{\delta_0}\right)^{1-\frac{2}{p}} \|x\|_q, \quad 2 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1;$$

$$\prod_{k=0}^{n-1} \mathcal{S}(\mathcal{F}^k(x)) \geq \delta^n;$$

$$\sum_{k=0}^{n-1} H(|\mathcal{F}^k(x)|^2) \geq \|x\|_2 (n \log \delta - 2n \log \|x\|_2).$$

Tao's uncertainty principle

Theorem (Tao, 2005)

For the abelian group $\mathbb{Z}_p$, p prime,

$$|supp(f)| + |supp(\hat{f})| \geq p + 1.$$

Question

Suppose $\mathscr{P}$ is a finite depth subfactor planar algebra with index p for some prime number p. Is the following inequality

$$\mathcal{S}(x) + \mathcal{S}(\mathcal{F}(x)) \geq p + 1 \tag{3}$$

true for a non-zero 2-box x?

Remarks

Remark

Our method is based on subfactor planar algebras. We combined its categorial and analytic properties. The method also works for (C^* spherical) planar algebras with multiple kinds of regions and strings.

Further Researches:

- Applications
- Infinite index
- n-box spaces

Thank you!

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