Classification of Spherical Quadrilaterals

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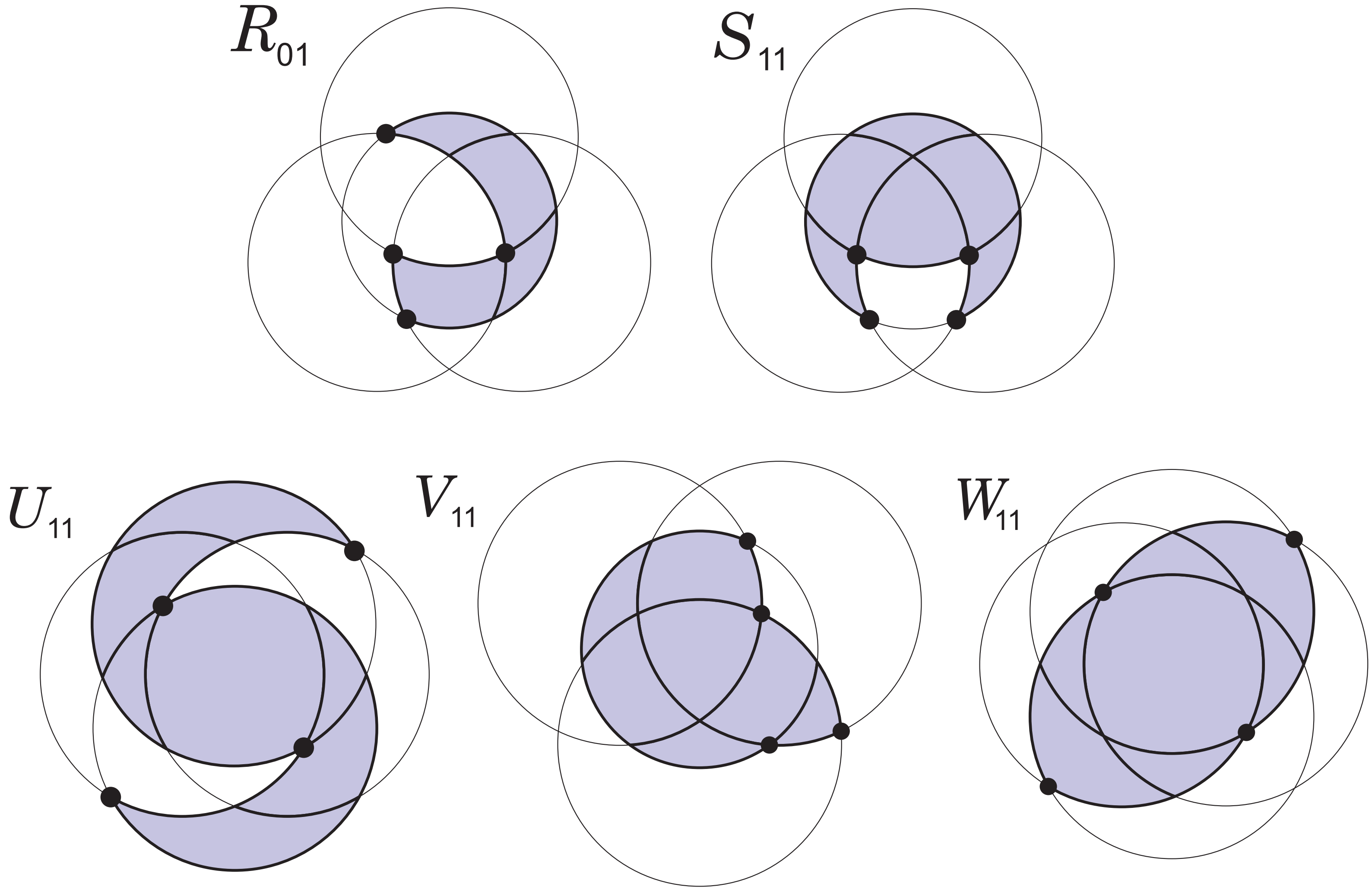

R_{01}
S_{11}
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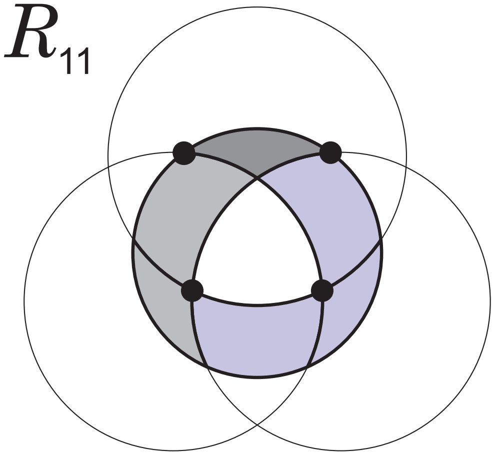
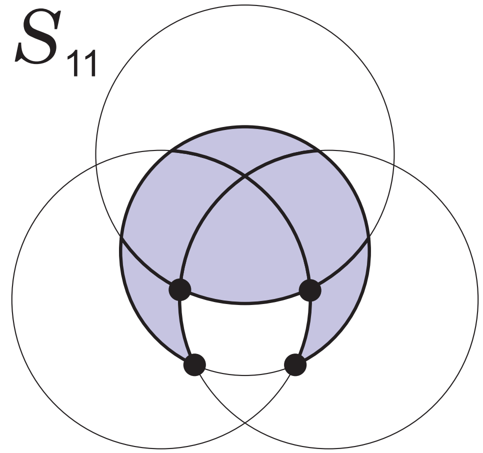
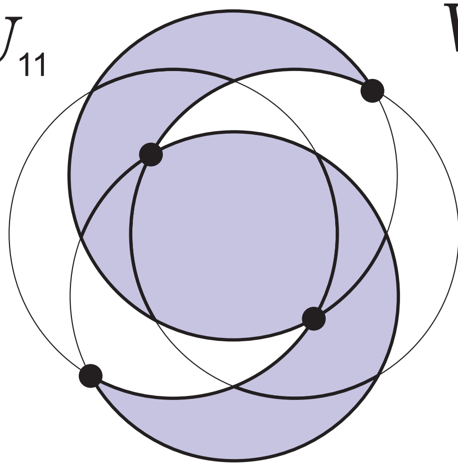
R_{11}



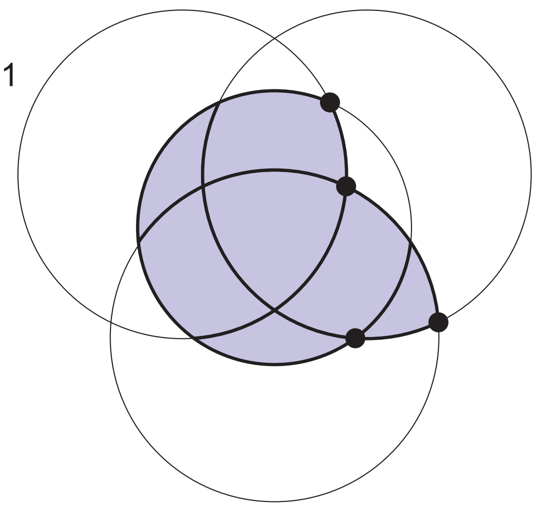
S_{11}



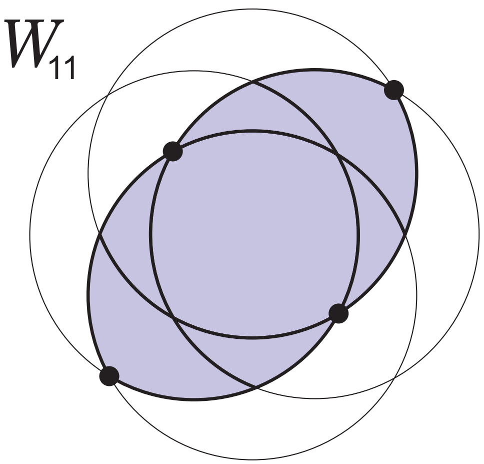
U_{11}



V_{11}



W_{11}



A **spherical polygon** is a surface homeomorphic to the closed disk, with several marked points on the boundary called **corners**, equipped with a Riemannian metric of constant curvature $K = 1$, such that the **sides** (arcs between the corners) are geodesic, and the metric has **conical singularities** at the corners.

A conical singularity is a point near which the length element of the metric is

$$ds = \frac{2\alpha|z|^{\alpha-1}|dz|}{1+|z|^2},$$

where z is a local conformal coordinate. The number $2\pi\alpha > 0$ is the angle at the conical singularity. The interior angle of our polygon is $\pi\alpha$. These angles can be arbitrarily large.

Every polygon can be mapped conformally onto the unit disk. We consider the problem of classification up to isometry of polygons with prescribed angles and prescribed corners.

By **prescribed corners** we mean that the images of the corners on the unit circle are prescribed.

The necessary condition on the angles,

$$\sum \alpha_j > n - 2,$$

follows from the Gauss–Bonnet formula. If $0 < \alpha_j < 1$, then we have existence and uniqueness (M. Troyanov, 1991, F. Luo and G. Tian, 1992).

Spherical triangles were classified by F. Klein, 1890, A. Eremenko, 2004, S. Fujimori *et al*, 2011.

If all α_j are not integers, the necessary and sufficient condition for the existence of a spherical triangle is

$$\cos^2\alpha_0 + \cos^2\alpha_1 + \cos^2\alpha_2 + 2\cos\alpha_0\cos\alpha_1\cos\alpha_2 < 1,$$

and the triangle is unique.

If α_0 is an integer, then the necessary and sufficient condition is that either $\alpha_1 + \alpha_2$ or $\alpha_1 - \alpha_2$ is an integer $m < \alpha_0$, with m and α_0 of opposite parity.

The triangle with an integer corner is not unique: there is a 1-parametric family when only one angle is integer, and a 2-parametric family when all angles are integer.

Developing map. A surface D of constant curvature 1 is locally isometric to a region on the standard sphere $\mathbf{S}$. This isometry is conformal, has an analytic continuation to the whole polygon, and is called the **developing map** $f : D \to \mathbf{S}$.

We say that spherical polygons are **equivalent** if their developing maps differ by a post-composition with a fractional-linear transformation.

Let us choose the upper half-plane H as the conformal model of our polygon, with n corners $a_0, \ldots, a_{n-1}$, and choose $a_{n-1} = \infty$. Accordingly, we sometimes denote α_{n-1} as α_∞. The other corners are real numbers.

Then $f : H \to \mathbf{S}$ is a meromorphic function mapping the sides into great circles. By the Symmetry Principle, f has an analytic continuation to a multi-valued function in $\overline{\mathbf{C}} \setminus \{a_0, \ldots, a_{n-1}\}$ whose monodromy is a subgroup of $PSU(2)$.

Such a function must be a ratio of two linearly independent solutions of the Fuchsian differential equation

$$w'' + \sum_{k=0}^{n-2} \frac{1-\alpha_k}{z-a_k}\, w' + \frac{P(z)}{\prod(z-a_k)}\, w = 0,$$

where P is a real polynomial of degree $n-3$ whose top coefficient can be expressed in terms of the α_j. The remaining $n-3$ coefficients of P are called the **accessory parameters**. The monodromy group of this equation must be conjugate to a subgroup of $PSU(2)$.

In the opposite direction, if a Fuchsian differential equation with real singularities and real coefficients has the monodromy group conjugate to a subgroup of $PSU(2)$, then the ratio of two linearly independent solutions restricted to H is a developing map of a spherical polygon.

Thus classification of spherical polygons with given angles and corners is equivalent to the following problem:

For a Fuchsian equation with given real parameters a_j, α_j*, to find the real values of accessory parameters for which the monodromy group of that equation is conjugate to a subgroup of* $PSU(2)$*. These values of accessory parameters are in bijective correspondence with the equivalence classes of spherical polygons.*

Spherical polygons with all integer angles. In this case, the developing map is a real rational function with real critical points. The multiplicities of the critical points are $\alpha_j - 1$. Such functions have been studied in great detail (A. Eremenko and A. Gabrielov, 2002, 2011, I. Scherbak, 2002, A. Eremenko, A. Gabrielov, M. Shapiro, F. Vainshtein, 2006).

The necessary and sufficient condition on the angles is $\sum(\alpha_j - 1) = 2d - 2$, where $d = \deg f$ is an integer, and $\alpha_j \leq d$ for all j. For given angles, there exist exactly $K(\alpha_0 - 1, \ldots, \alpha_{n-1} - 1)$ of the equivalence classes of polygons, where K is the **Kostka number**: it is the number of ways to fill in a table with two rows of length $d - 1$ with $\alpha_0 - 1$ zeros, $\alpha_1 - 1$ ones, *etc.*, so that the entries are non-decreasing in the rows and increasing in the columns.

Polygons with two non-integer angles. Let α_0 and α_{n-1} be non-integer, while the rest of the angles α_j are integer.

Assuming $a_0 = 0$ and $a_{n-1} = \infty$ we conclude that the developing map has the form

$$f(z) = z^{\alpha}\,\frac{P(z)}{Q(z)},$$

where $\alpha \in (0,1)$ and P, Q are real polynomials.

For this case, a necessary and sufficient condition on the angles is the following

Theorem 1. *Let*

$$\sigma := \alpha_1 + \ldots + \alpha_{n-2} - n + 2.$$

a) If $\sigma + [\alpha_0] + [\alpha_{n-1}]$ is even, then $\alpha_0 - \alpha_{n-1}$ is an integer of the same parity as σ, and

$$|\alpha_0 - \alpha_{n-1}| \leq \sigma.$$

b) If $\sigma + [\alpha_0] + [\alpha_{n-1}]$ is odd, then $\alpha_0 + \alpha_{n-1}$ is an integer of the same parity as σ, and

$$\alpha_0 + \alpha_{n-1} \leq \sigma.$$

Finding all polygons with prescribed angles is equivalent in this case to solving the equation

$$z(P'Q - PQ') + \alpha PQ = R$$

with respect to real polynomials P and Q of degrees p and q, respectively, where R is a given real polynomial of degree $p+q$. The degree of the map

$$W_\alpha : (P, Q) \mapsto z(P'Q - PQ') + \alpha PQ$$

equals

$$\binom{p+q}{p}$$

(it is a linear projection of a Veronese variety), and one can show that when all roots of R are non-negative, all solutions $(P, Q) \in W_\alpha^{-1}(R)$ are real.

Enumeration of polygons with two adjacent non-integer angles. An important special case is when a_0 and a_{n-1} are adjacent corners of the polygon, $2\alpha_0$ and $2\alpha_{n-1}$ are odd integers, while all other α_j are integers. Equivalence classes of such polygons are in bijective correspondence with **odd real rational functions with all critical points real**, given by

$$f(z) = g(\sqrt{z}),$$

where f is the developing map of our polygon and g is a rational function as above.

By a deformation argument, this gives the following

Theorem 2. *If the angles satisfy the necessary and sufficient condition given above, and the corners* $a_0 = 0$ *and* $a_{n-1} = \infty$ *are adjacent, then there are exactly*

$$E(2\alpha_0 - 1, \alpha_1 - 1, \ldots, \alpha_{n-2} - 1, 2\alpha_{n-1} - 1)$$

equivalence classes of polygons, where $E(m_0, \ldots, m_{n-1})$ *is the number of chord diagrams in* H*, symmetric with respect to* $z \mapsto -\overline{z}$*, with the vertices* $0 = a_0 < a_1 < \ldots < a_{n-2} < a_{n-1} = \infty$ *and* $-a_1, \ldots, -a_{n-2}$*, and* m_j *chords ending at each vertex* a_j*.*

If a_0 and a_{n-1} are not adjacent, E gives an upper bound on the number of equivalence classes of polygons.

One can express E in terms of the Kostka numbers.

Proposition. *Let m_0 and m_{n-1} be even. Then*

$$E(m_0, m_1, \ldots, m_{n-2}, m_{n-1}) = K(r, m_1, \ldots, m_{n-2}, s),$$

where positive integers r and s satisfy

$$r + s > m_1 + \ldots + m_{n-2}, \tag{1}$$

and can be defined as follows:
If $\mu := (m_0 + m_{n-1})/2 + m_1 + \ldots + m_{n-2}$ is even, then $r = m_0/2 + k$, $s = m_{n-1}/2 + k$, where k is large enough, so that (1) is satisfied.
If μ is odd, then $r = (m_0 + m_{n-1})/2 + k + 1$, $s = k$, and k is large enough, so that (1) is satisfied.

Spherical quadrilaterals. Heun's equation. In the case $n = 4$ the Fuchsian equation for the developing map is the Heun's equation

$$w'' + \left(\frac{1-\alpha_0}{z} + \frac{1-\alpha_1}{z-1} + \frac{1-\alpha_2}{z-a}\right) w' + \frac{Az-\lambda}{z(z-1)(z-a)} w = 0,$$

where A can be expressed in terms of α_j, and λ is the accessory parameter.

We can place three singularities at arbitrary points, so we choose $a_0 = 0$, $a_1 = 1$, $a_2 = a$, $a_3 = \infty$.

The condition that the monodromy belongs to $PSU(2)$ is equivalent to an equation of the form $F(a, \lambda) = 0$. This equation is algebraic if at least one or the angles is integer.

Theorem 2 in the case of quadrilaterals with two integer and two non-integer angles specializes to the following

Theorem 3. *The number of classes of quadrilaterals with two integer and two non-integer angles is at most*

$$\min\{\alpha_1, \alpha_2, k+1\},$$

where

$$k+1 = \begin{cases} (\alpha_1 + \alpha_2 - |\alpha_0 - \alpha_3|)/2 & \textit{in case a)} \\ (\alpha_1 + \alpha_2 - \alpha_0 - \alpha_3)/2 & \textit{in case b).} \end{cases}$$

If $a > 0$ we have equality.

Here cases a) (when $\alpha_0 - \alpha_3$ is integer) and b) (when $\alpha_0 + \alpha_3$ is integer) are as in Theorem 1. Condition $a > 0$ means that the corners a_1 and a_2 with integer angles are adjacent.

Quadrilaterals with non-adjacent integer angles. Let $\delta = \max(0, \alpha_1 + \alpha_2 - [\alpha_0] - [\alpha_3])/2$.

Theorem 4. *The number of equivalence classes of quadrilaterals with non-adjacent corners a_1 and a_2, with integer angles α_1 and α_2, is at least*

$$\min\{\alpha_1, \alpha_2, k+1\} - 2\left[\frac{1}{2}\min\{\alpha_1, \alpha_2, \delta\}\right], \qquad (2)$$

where k is the same as in Theorem 3.

Notice that in case b) of Theorems 1 and 3, the lower bound (2) becomes 0 when $\min\{\alpha_1, \alpha_2, k+1\}$ is even and 1 if $\min\{\alpha_1, \alpha_2, k+1\}$ is odd.

Nets. The developing map is a local homeomorphism, except at the corners, of a closed disk D to the standard sphere $\mathbf{S}$. The sides are mapped to great circles. These great circles define a partition (cell decomposition) of the sphere. Taking the f-preimage of this partition, and adding vertices corresponding to the integer corners, we obtain a cell decomposition of D which is called a **net.** Two nets are considered equivalent if they can be mapped to each other by an orientation-preserving homeomorphism of the disk, respecting labeling of the corners.

It is easy to see that a net, together with the partition of the sphere by the great circles, define the polygon completely.

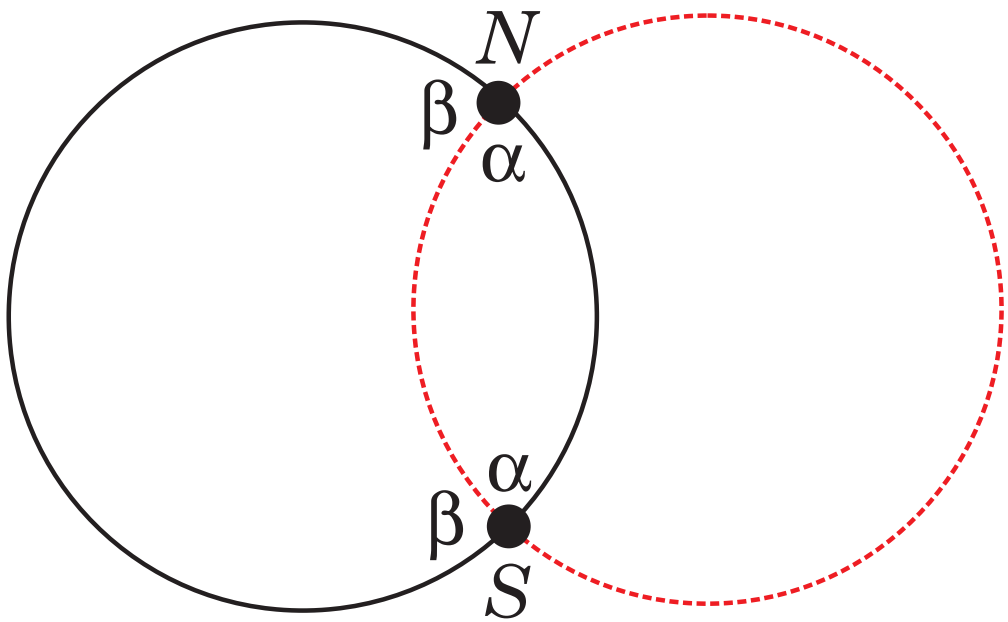


Fig. 1. Partition of the Riemann sphere by two great circles.

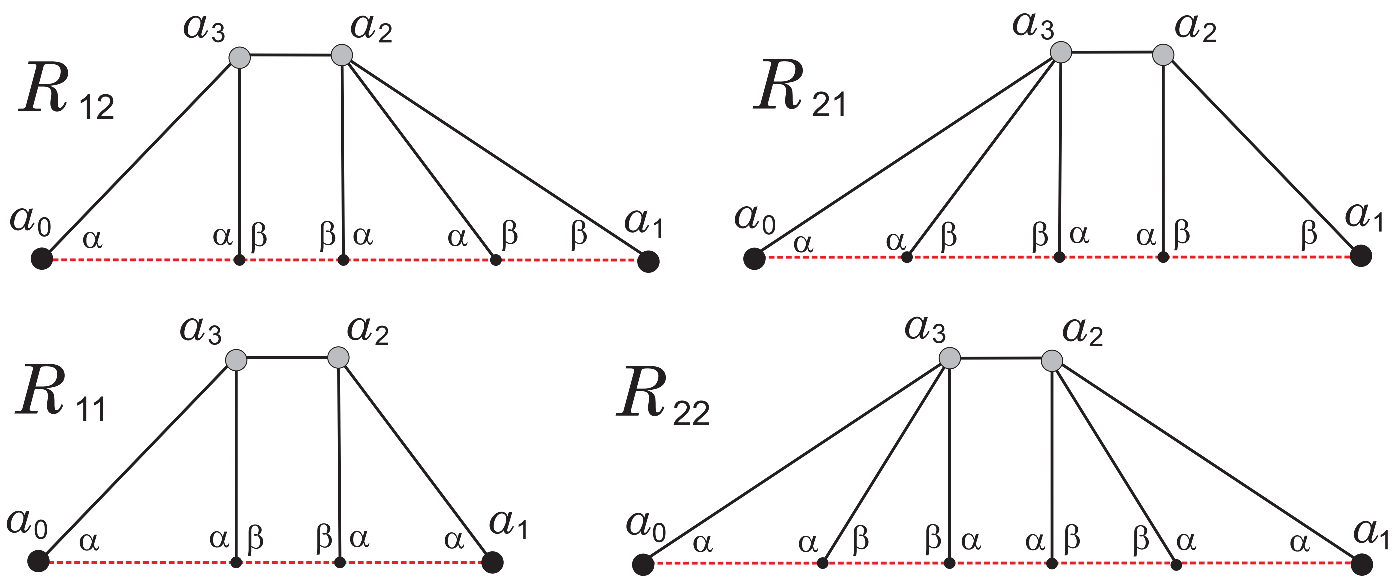


Fig. 2. Primitive nets, two adjacent integer corners.

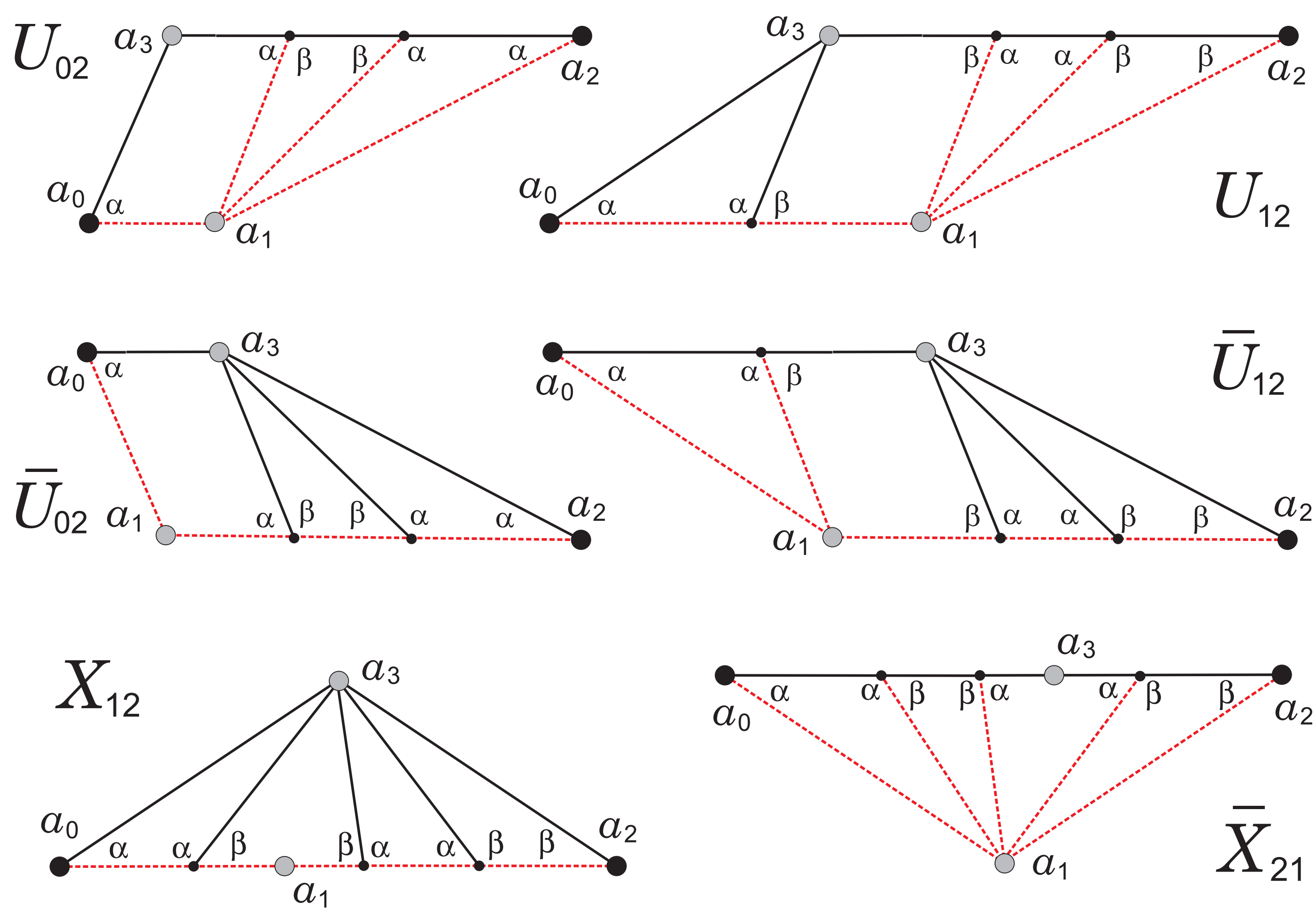


Fig. 3. Primitive nets, two opposite integer corners.

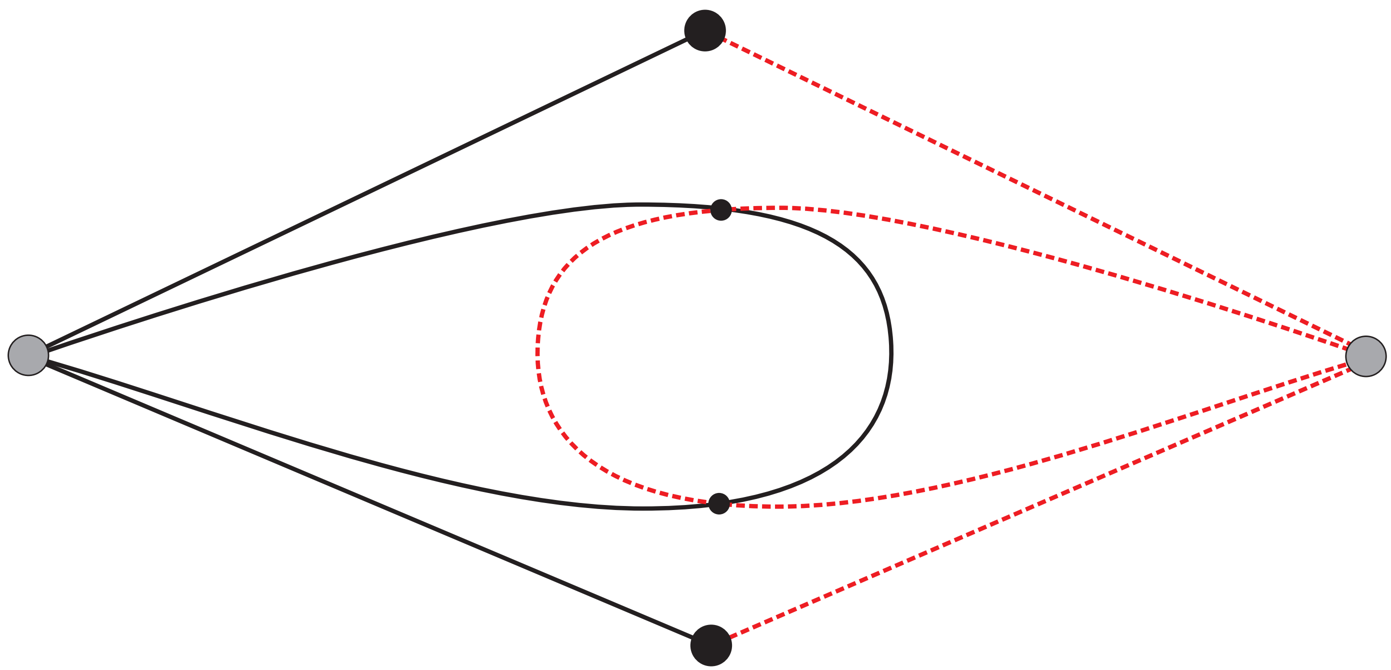


Fig. 4. Pseudo-diagonal, two opposite integer corners.

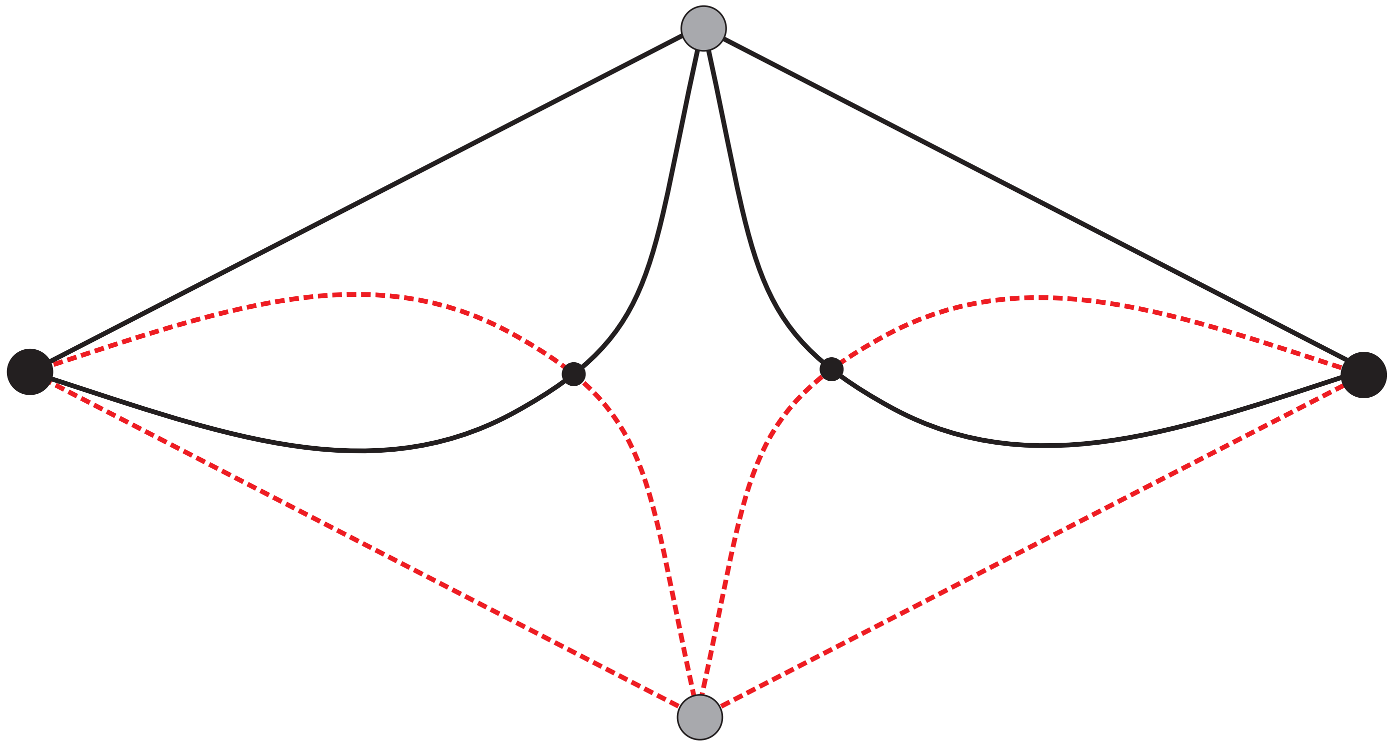


Fig. 5. Non-uniqueness, two opposite integer corners.

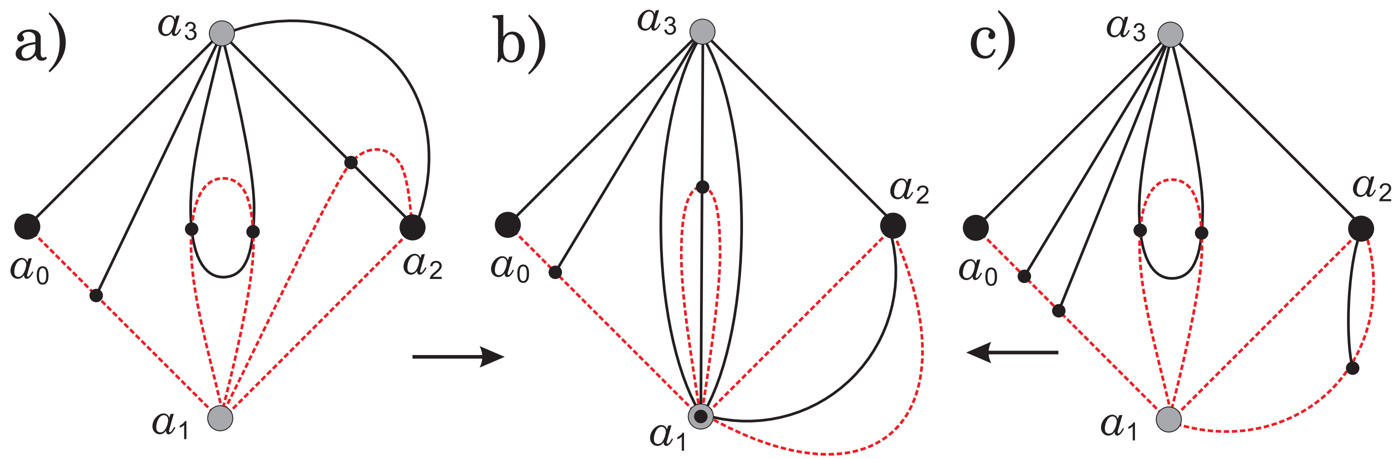


Fig. 6. A chain of nets, two opposite integer corners.

Quadrilaterals with three non-integer angles.

Suppose that α_3 is integer while the rest of the angles are not. The necessary and sufficient condition for the existence of a quadrilateral with the given angles is the same as in the case of triangles, and the number of quadrilaterals with the given angles is at least

$$\alpha_3 - 2\left[\min\left(\frac{\alpha_3}{2}, \frac{[\alpha_1]+1}{2}, \frac{\delta+1}{4}\right)\right]$$

where $\delta = \max(0, [\alpha_1] + \alpha_3 - [\alpha_0] - [\alpha_2])$.

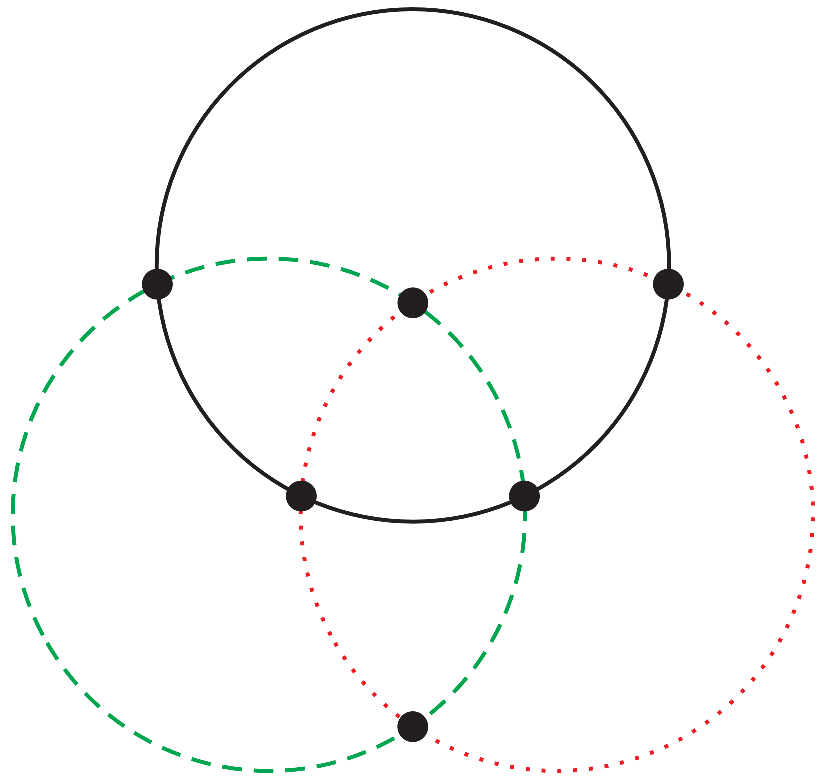


Fig. 7. Partition of the Riemann sphere by three great circles.

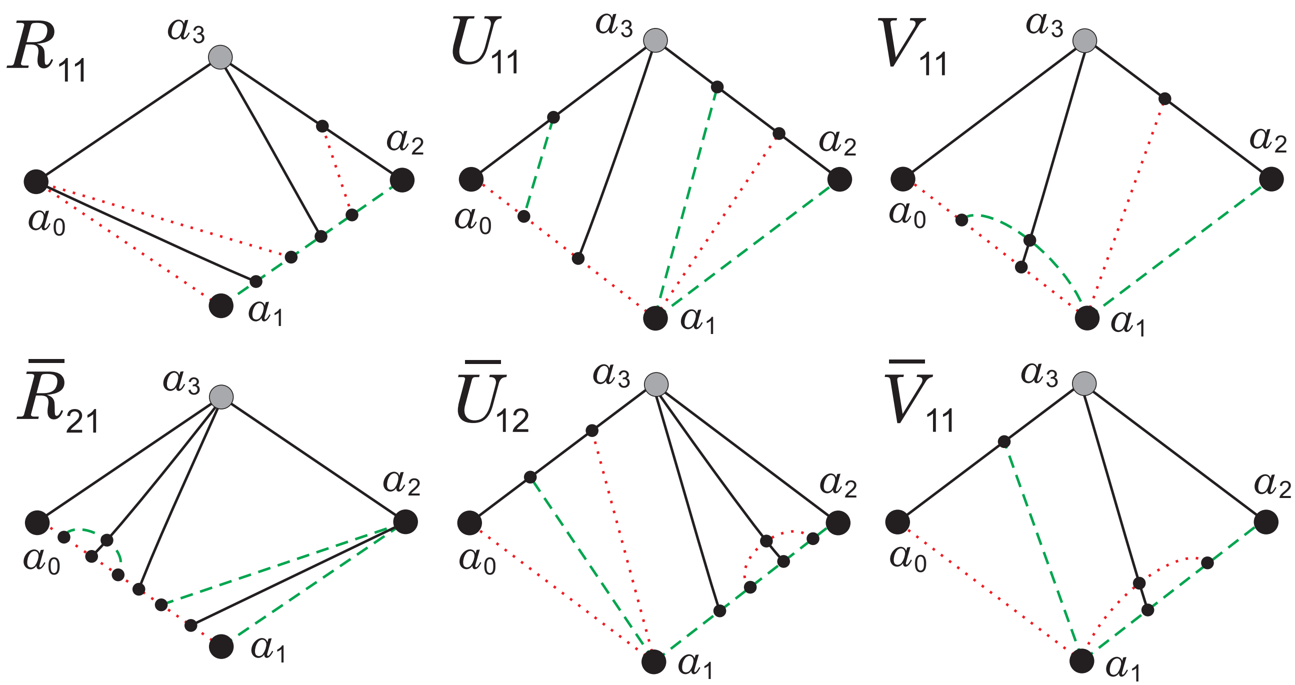


Fig. 8. Primitive nets, three non-integer corners.

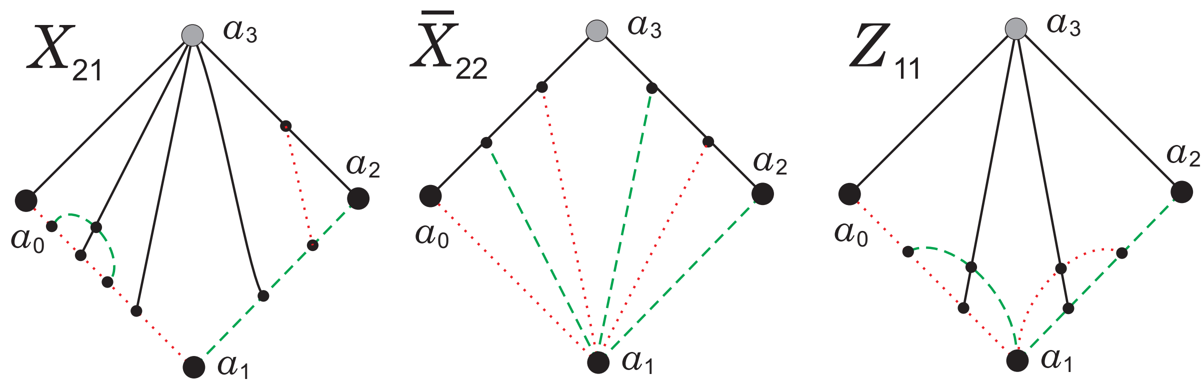


Fig. 9. Primitive nets, three non-integer corners.

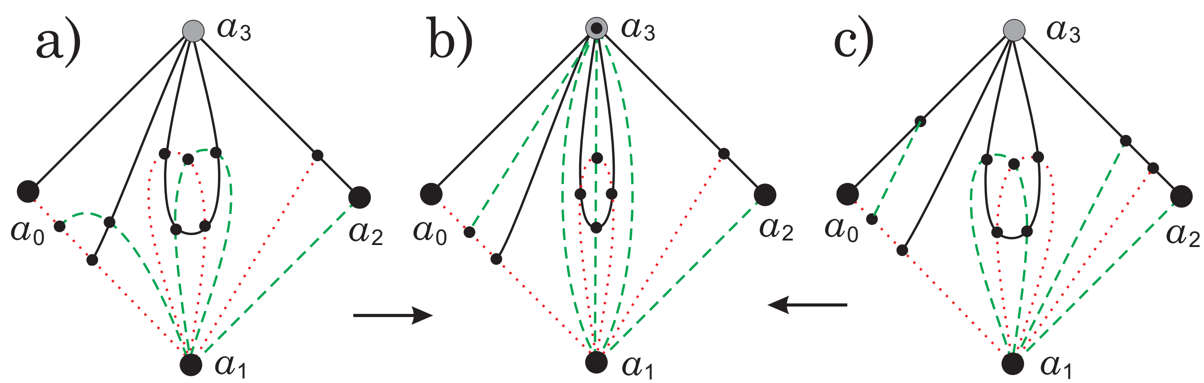


Fig. 10. A chain of nets, three non-integer corners.

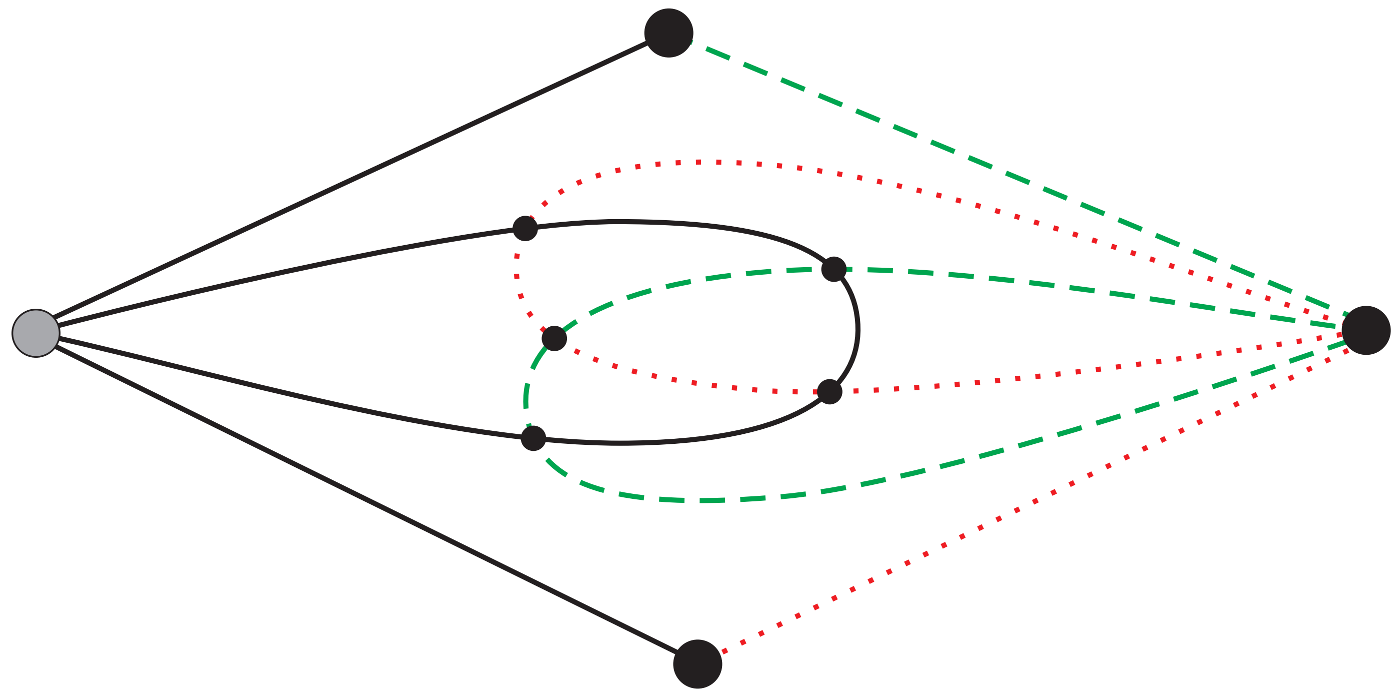


Fig. 11. Pseudo-diagonal, three non-integer corners.

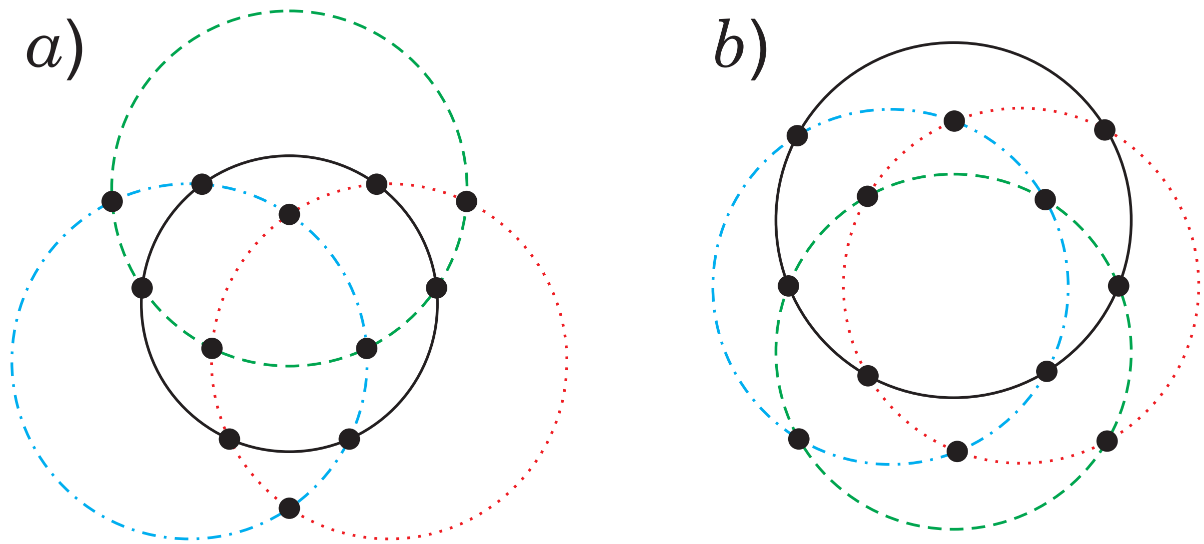


Fig. 12. Partition of the Riemann sphere by four great circles (two views).

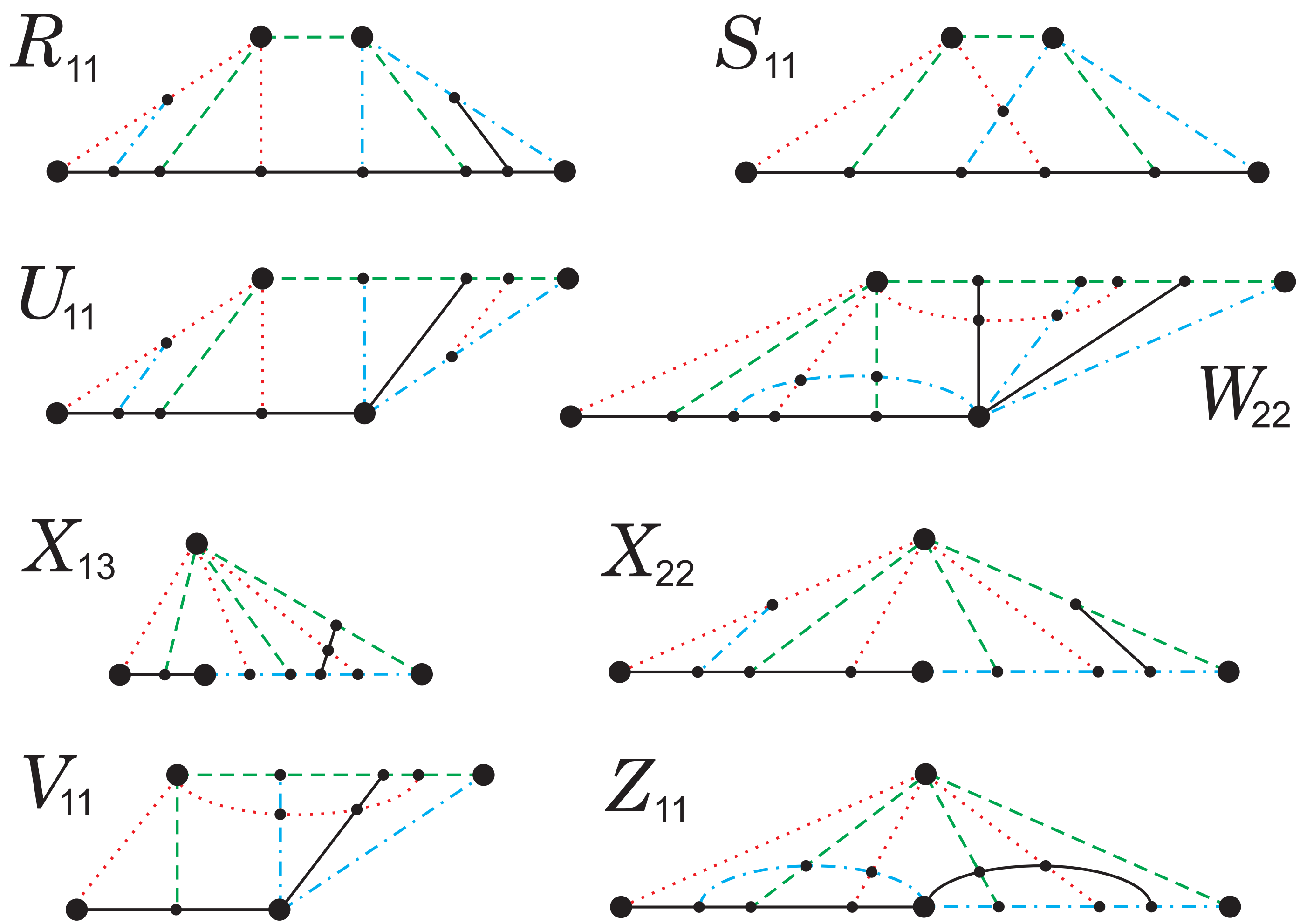


Fig. 13. Primitive nets, four non-integer corners.

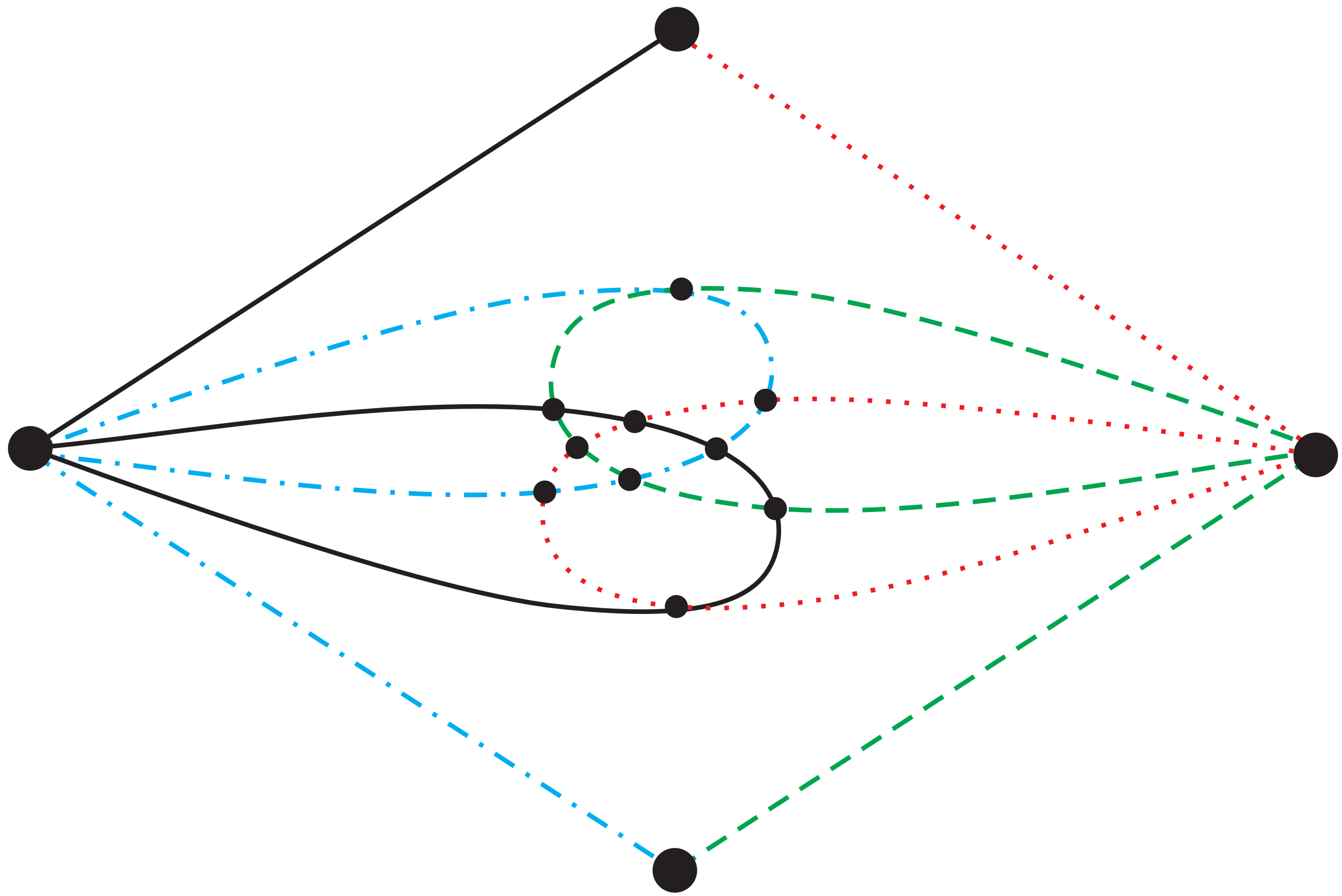


Fig. 14. Pseudo-diagonal, four non-integer corners.

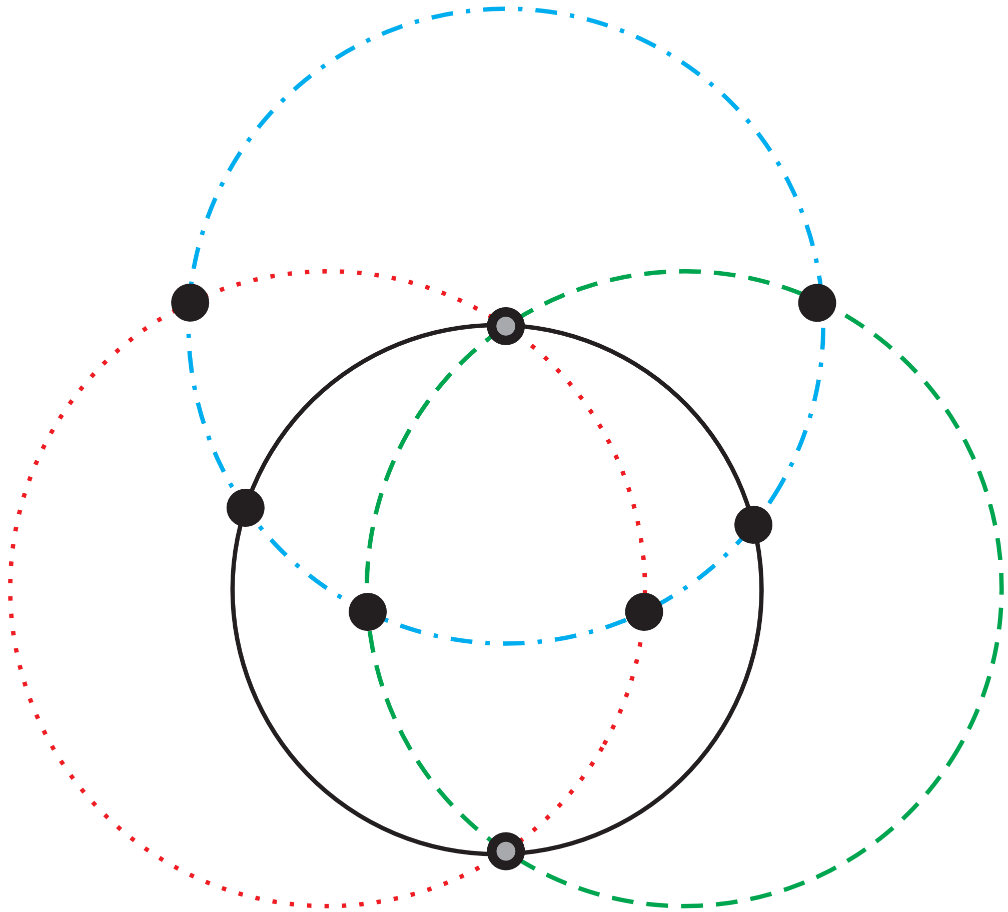


Fig. 15. Partition of the Riemann sphere by non-generic four great circles.

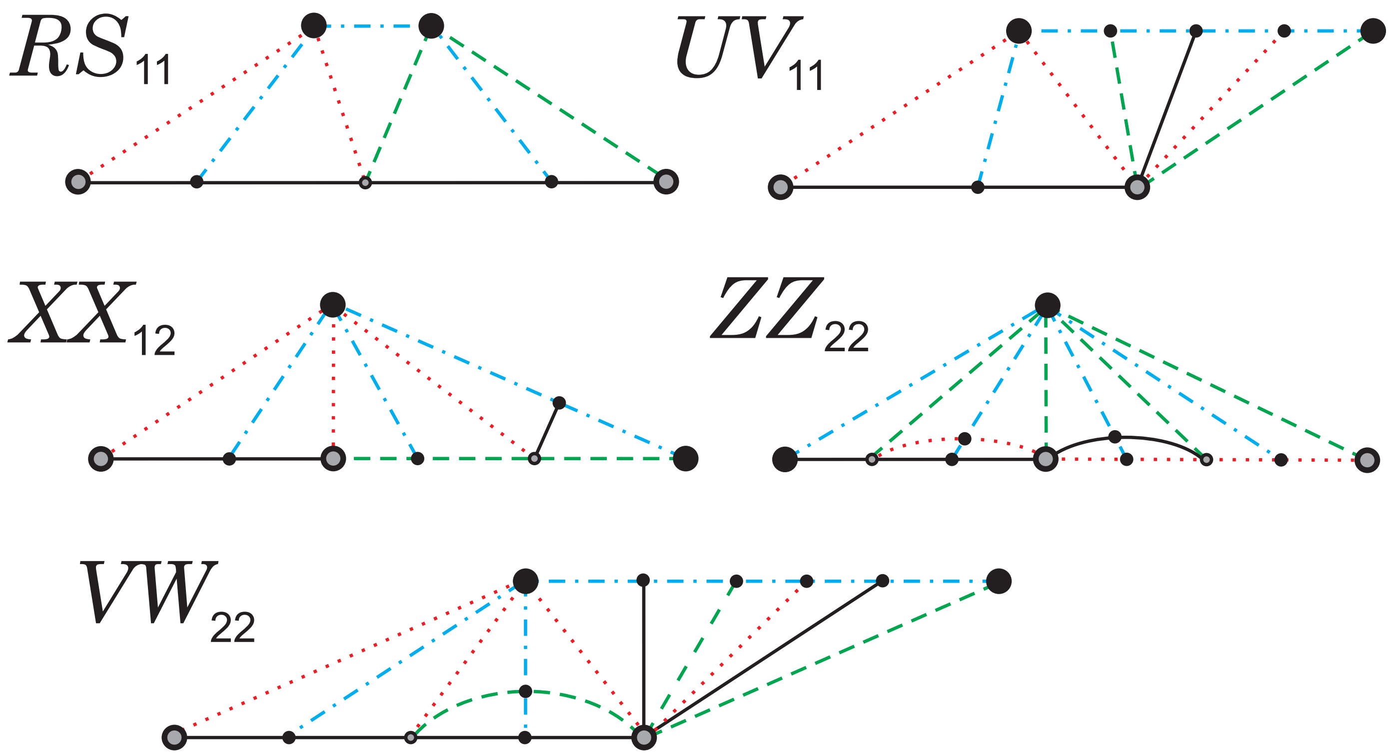


Fig. 16. Some nets for non-generic four great circles.

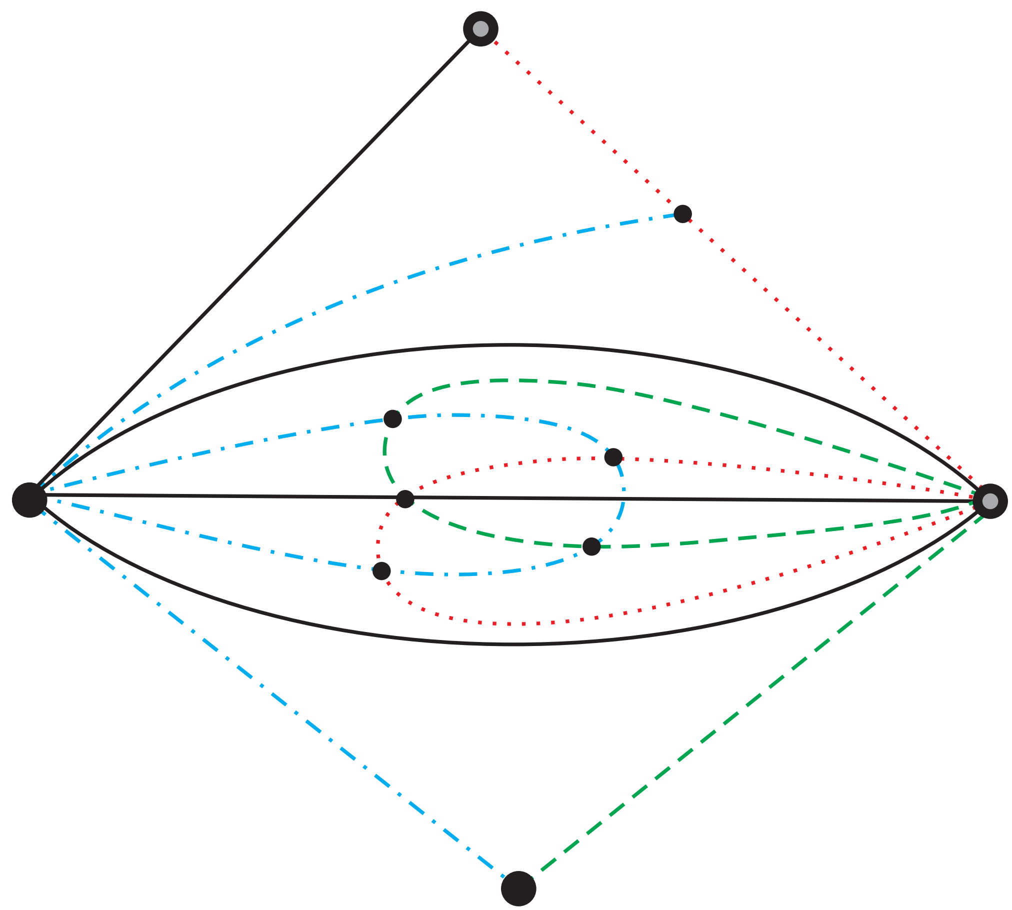


Fig. 17. Pseudo-diagonal for non-generic four great circles.

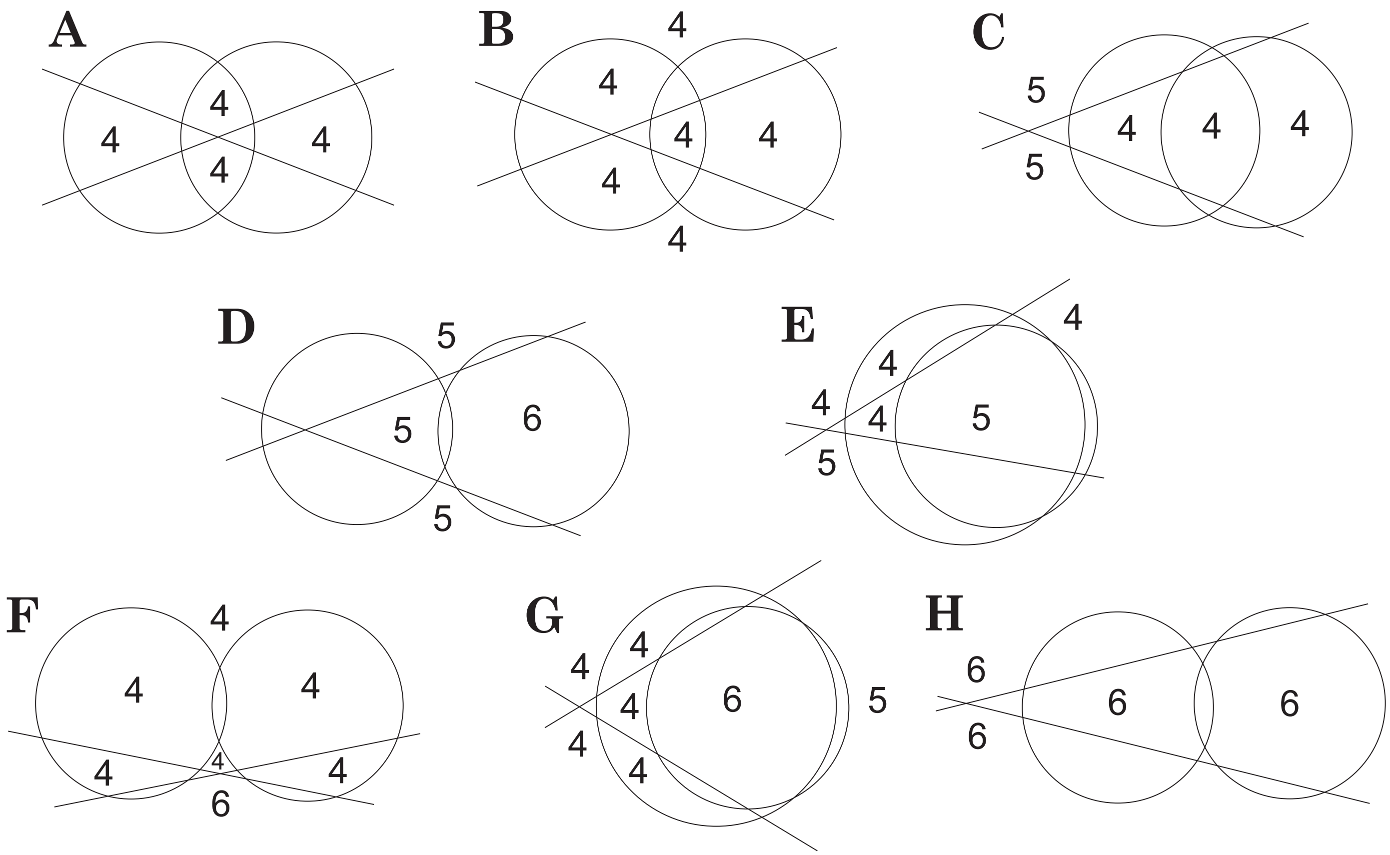


Fig. 18. Partitions of the Riemann sphere by four non-geodesic circles.