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Hyperkähler Surjectivity

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Hyperkähler manifolds

Definition

A hyperkähler manifold is a manifold M equipped with three symplectic structures $\omega_1, \omega_2, \omega_3$. These are organized as $\omega_{\mathbb{R}} = \omega_1$ (real moment map) and $\omega_{\mathbb{C}} = \omega_2 + i\omega_3$ (complex moment map).

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Hyperkähler quotient: If a compact Lie group G acts on M and the action is Hamiltonian with respect to all three symplectic structures, (with moment maps μ_1, μ_2, μ_3) then the hyperkähler quotient is defined as $M///G = (\mu_{HK})^{-1}(0)/G$ where $\mu_{HK} = (\mu_1, \mu_2, \mu_3)$. (by analogy with the Kähler quotient $M//G := \mu^{-1}(0)/G$) where μ is the moment map).

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HK quotients are closely related to problems in gauge theory (instantons, for example the ADHM construction) and string theory (supersymmetric sigma models).

Examples

Hypertoric varieties are hyperkähler analogues of toric varieties, and in particular their holomorphic symplectic structures are completely integrable (Bielawski-Dancer, Konno, Hausel-Sturmfels)

Hyperpolygon spaces are hyperkähler analogues of moduli spaces of euclidean *n*-gons, and are related to certain Hitchin systems on $\mathbb{CP}^1$ (Konno, Hausel-Proudfoot, Harada-Proudfoot, Godinho-Mandini, Fisher-Rayan)

Nakajima quiver varieties are hyperkähler manifolds associated to quivers, used to construct moduli spaces of Yang-Mills instantons as well as representations of Kac-Moody algebras (Atiyah-Hitchin-Drinfeld-Manin, Kronheimer, Nakajima)

Definition

- Suppose M is a symplectic manifold equipped with Hamiltonian G action. The *Kirwan map* is the map (where H_G^* denotes equivariant cohomology).

$$\kappa : H_G^*(M) \to H_G^*\big(\mu^{-1}(0)\big) \cong H^*(\mu^{-1}(0)/G)$$

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- When M is compact, Kirwan proved that this map is surjective.
- Hyperkähler Hamiltonian actions never exist on compact HK manifolds though.
- The hyperkähler Kirwan map is defined as

$$\kappa^{HK} : H_G^*(M) \to H^*\big(\mu_{HK}^{-1}(0)/G\big)$$

  where $\mu_{HK} = (\mu_1, \mu_2, \mu_3)$.

The Kirwan map

Our theorem is

Theorem

For a large class of Hamiltonian hyperkähler manifolds (those of linear type)

- *The hyperkähler Kirwan map is surjective, except possibly in middle degree.*
- *The natural restriction* $H^i(M//G) \to H^i(M///G)$ *is an isomorphism below middle degree and an injection in middle degree.*

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The second point means that the kernel (and hence image) of the hyperkähler Kirwan map can be computed using standard techniques.

Definition

M is circle compact if it is equipped with a Hamiltonian S^1 action for which

1. The fixed point set is compact
2. The S^1 moment map is proper and bounded below

Definition

A G-action on a hyperkähler manifold M is said to be of *linear type* if the following conditions are satisfied:

- M is circle compact and the S^1-action commutes with the G-action.
- Both $M/\!/G$ and $M/\!/\!/G$ are circle compact with respect to the induced S^1-actions.
- The holomorphic symplectic form $\omega_{\mathbb{C}}$ and complex moment map $\mu_{\mathbb{C}}$ are homogeneous of positive degree with respect to the S^1-action, i.e. $\phi_t^*\omega_{\mathbb{C}} = t^d\omega_{\mathbb{C}}$ and $\mu_{\mathbb{C}} \circ \phi_t = t^d\mu_{\mathbb{C}}$ for some $d > 0$, where ϕ_t denotes the S^1-action map.
- $\overline{M}$ is smooth and the line bundle $L_{\overline{M}}(D_M)$ is ample on $\overline{M}$.

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Hyperkähler surjectivity was already known for hypertoric varieties and hyperpolygon spaces (Konno). It is known for quiver varieties only in certain special cases.

Definition

The *cut compactification* of a circle compact manifold M is the manifold

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where c is a large real number. The boundary divisor is $\overline{M} \smallsetminus M$.

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Lemma

If M is circle compact, then the natural restriction $H^*(\overline{M}) \to H^*(M)$ is surjective.

Proof.

We have $\overline{M}^{S^1} = M^{S^1} \sqcup D_M$. It follows immediately from Morse theory that we have the short exact sequence

$$0 \longrightarrow H^{*-2}_{S^1}(D_M) \longrightarrow H^*_{S^1}(\overline{M}) \longrightarrow H^*_{S^1}(M) \longrightarrow 0$$

The statement in ordinary cohomology then follows by equivariant formality. $\square$

Remark

If $\bar{M}$ is smooth, we have a Thom-Gysin sequence

$$\cdots \to H^{i-2}(D_M) \to H^i(\overline{M}) \to H^i(M) \to \dots$$

Theorem

If M is a hyperkähler manifold with a G-action of linear type, then the Kirwan map $\kappa : H^*_G(M) \to H^*(M/\!/G)$ is surjective.

Proof.

Consider the inclusion of $M \times \mathbb{C}^*$ into $M \times \mathbb{C}$. We have

$$\begin{array}{ccc} H^*_{G\times S^1}(M \times \mathbb{C}) & \longrightarrow & H^*(\overline{M/\!/G}) \\ \downarrow & & \downarrow \\ H^*_{G\times S^1}(M \times \mathbb{C}^*) & \longrightarrow & H^*(M/\!/G) \end{array}$$

The right vertical arrow is surjective by the previous Lemma. The top horizontal arrow is also surjective (by usual Atiyah-Bott-Kirwan theory). The result follows because the S^1 action on $M \times \mathbb{C}^*$ is free, so $H^*_{G\times S^1}(M \times \mathbb{C}^*) \cong H^*_G(M)$. □

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- *Then the natural restriction $H^i(\overline{M/\!/\!/G}) \to H^i(M/\!/\!/G)$ is an isomorphism below middle degree and an injection in middle degree.*
- *Furthermore, $H^i(M/\!/\!/G)$ vanishes above middle degree. Consequently, the hyperkähler Kirwan map is surjective except possibly in middle degree, and its kernel is generated by $\ker\left(H^*_G(M) \to H^*(M/\!/G)\right)$ together with all classes above middle degree.*

Examples where surjectivity is known even in middle degree: hyperpolygon spaces (Konno), hypertoric manifolds (Konno), torus quotients of cotangent bundles of compact varieties (Fisher-Rayan 2014), Hilbert schemes of points on $\mathbb{C}^2$, Hilbert schemes of points on hyperkähler ALE spaces, moduli space of rank 2 odd degree Higgs bundles.

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Using very different techniques, McGerty and Nevins have an apparently stronger surjectivity result but their proof does not give any information about the kernel of the HK Kirwan map

Proof of our main theorem.

The proof follows Bott's proof of the Lefschetz hyperplane theorem (1959, using Morse theory), working on the cut compactification $\overline{M}$ (which is assumed to be smooth). It is analogous to Sommese's theorem for ample vector bundles.

Bott's argument is applied to the logarithm of a product of components of the moment map, making use of an inductive argument on restriction to intersections of subsets (induction on the number of subsets).

$\square$

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- Hence we obtain the commutative diagrams

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- By our earlier result the middle vertical arrow is an isomorphism for i below middle degree (and an injection in middle degree).
- We also have that the left vertical arrow is an isomorphism (for the same range of i). Hence so is the restriction $H^i(M//G) \to H^i(M///G)$.