# Linear ODEs from an algebraic point of view

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**Conference**

**Legacy of Vladimir Arnold**

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We solve the ***generic*** order $r$ linear ODE

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$$\sum_{j \in \mathbb{Z}} h_j z^j = \frac{1}{1 - e_1 z + e_2 z^2 - \ldots + (-1)^r e_r z^r}.$$

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*$h_j,\ j \in \mathbb{Z}$, the complete sym functions:* $h_j = 0\ \ (j < 0)$,

$h_0 = 1, h_1 = e_1, h_2 = e_1^2 - e_2, \ldots$

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$$X(t) = \exp\left(M_r t\right) \cdot C, \quad C = \begin{pmatrix} c_0 \\ \vdots \\ c_{r-1} \end{pmatrix} = X(0)$$

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<u>Remark</u>: $\exp\left(M_r t\right)$ is the Wronski matrix of the *<u>standard</u>* fundamental system of the solutions to the ODE, $\boxed{v_0(t), \ \cdots \ , \ v_{r-1}(t)}$ , that is $v_i^{(j)}(0) = \delta_{ij}, \ 0 \le i, j \le r-1$ (standard initial conditions).

Theorem 2: The last column of $\exp\left(M_r t\right)$ is our *universal* fundamental system:

$$\exp\left(M_r t\right) = \left(\begin{array}{cccc} v_0 & v_1 & \cdots & v_{r-1} = u_{1-r} \\ v_0' & v_1' & \cdots & v_{r-1}' = u_{2-r} \\ \vdots & \vdots & \ddots & \vdots \\ v_0^{(r-1)} & v_1^{(r-1)} & \cdots & v_{r-1}^{(r-1)} = u_0 \end{array}\right)$$

$$u_0(t) = \sum_{k\geq 0} h_k \frac{t^k}{k!}, \ \ldots, \ u_{1-r}(t) = \sum_{k\geq r-1} h_{k+1-r} \frac{t^k}{k!}$$

The universal and the standard fundamental systems are related as follows,

$$\begin{pmatrix} u_0 \\ u_{-1} \\ \vdots \\ u_{1-r} \end{pmatrix} = \begin{pmatrix} 1 & h_1 & h_2 & \cdots & \cdots & h_r \\ 0 & 1 & h_1 & \cdots & \cdots & h_{r-1} \\ 0 & 0 & 1 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix} = \begin{pmatrix} 1 & -e_1 & e_2 & \cdots & \cdots & (-1)^r e_r \\ 0 & 1 & -e_1 & \cdots & \cdots & (-1)^{r-1} e_{r-1} \\ \vdots & 0 & 1 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_{-1} \\ \vdots \\ u_{1-r} \end{pmatrix}$$

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Generating functions of the complete and the elementary symmetric functions,

$H(z) = \sum_{k\geq 0} h_k z^k = \prod_{i\geq 1}(1-\xi_i z)^{-1}$ and $E(z) = \sum_{k=0}^{r} e_k z^k = \prod_{i=1}^{r}(1+\xi_i z)$

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$$\boxed{H(-z)E(z)=1}$$

Corollary: The (unique) solution to the *Cauchy problem* ,

$$u^{(r)} - e_1 u^{(r-1)} + e_2 u^{(r-2)} - \ldots + (-1)^r e_r u = 0, \;\; u^{(j)}(0) = c_j \in B_r,$$

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Solution in the <u>non-homogeneous case</u>, $g(t) = \sum_{k \geq 0} b_k \frac{t^k}{k!} \in B_r[[t]]$ :

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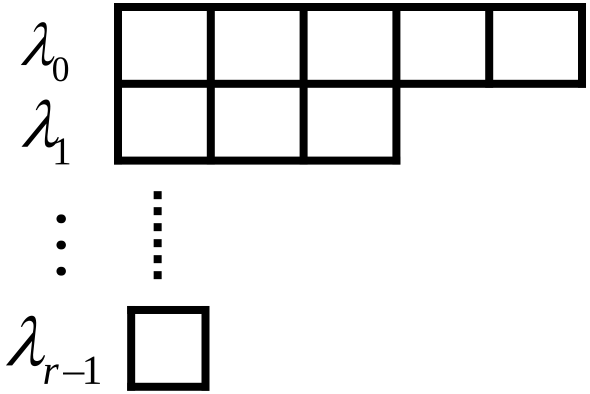
Theorem 3: We have $p(t) = \sum_{k \geq r} p_k \frac{t^k}{k!}$ , where $p_k \in B_r$, $k \geq r$, are given by

$$\sum_{k \geq r} p_k z^{k-r} = \frac{\sum_{k \geq 0} b_k z^k}{1 - e_1 z + e_2 z^2 - \ldots + (-1)^r e_r z^r} .$$




Young diagram of $\lambda$

## Schur function:

$$S_\lambda(\xi) = \det(\xi_{\lambda_j+i-j})_{0\le i,j\le r-1} \ ,$$

*where*

$$\lambda = (\lambda_0 \ge \lambda_1 \ge \ldots \ge \lambda_{r-1} \ge 0) \in P_r ,$$

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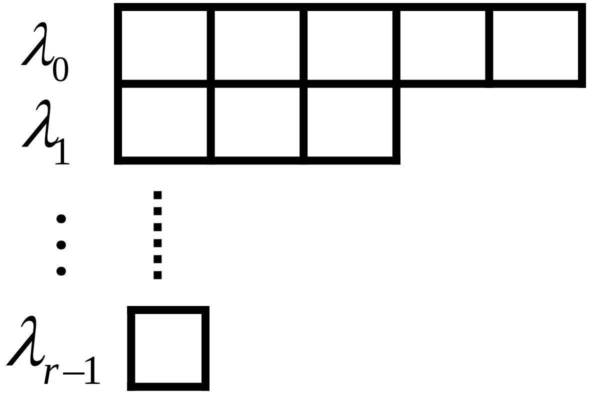

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<u>Examples:</u> The coefficients $e_1,\ldots,e_r$ of the ODE and the coefficients $h = \{h_j\}_{j\in\mathbb{Z}}$ of the universal solution are related by the *<u>Giambelli formula</u>*

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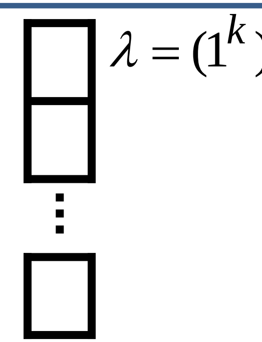

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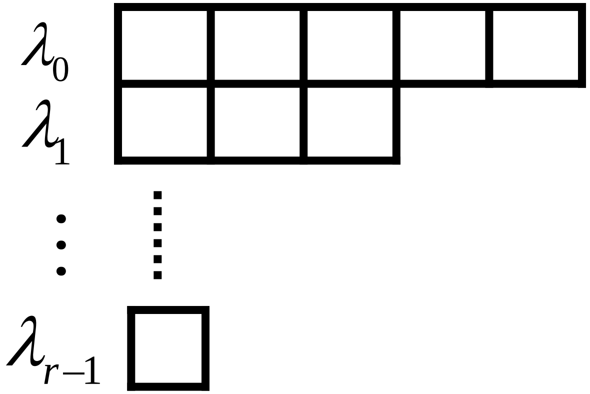


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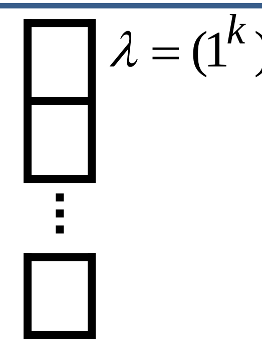


As the elementary and the complete symmetric functions

in $\xi_1, \xi_2, \ldots, \xi_r$ , $\boxed{e_k = S_{(1^k)}(\xi) \ , \ h_k = S_{(k)}(\xi)}$ , where

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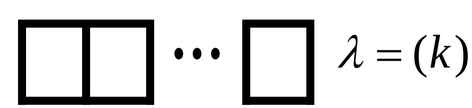

For $f_j = f_j(t) \in B_r[[t]]$ $(0 \le j \le r-1)$, denote $\overline{f} = (f_0, \; f_1, \dots, \; f_{r-1})$.

The Wronskian of $\overline{f}$: $\qquad W[\overline{f}](t) = \det(f_j^{(i)})_{0 \le i, j \le r-1}$.

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Motivation: The role Wronskians play in Schubert calculus on Grassmannian

[papers of L. Goldberg; A. Eremenko and A. Gabrielov; B. and M. Shapiro;

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<u>*Generalized Wronskian*</u> of $\overline{f}$ corresponding to $\lambda = (\lambda_0 \ge \lambda_1 \ge \ldots \ge \lambda_{r-1} \ge 0)$ :

$$W[\lambda, \overline{f}](t) = \det(f_j^{(\lambda_{r-1-i}+i)})_{0 \le i, j \le r-1}.$$

<u>Theorem 4</u> (Derivatives of the Wronskian in terms of the generalized Wronskians):

$$W[\overline{f}\,]^{(k)}(t) = \sum_{|\lambda|=k} c_\lambda W[\lambda, \overline{f}\,](t), \quad \text{where} \quad c_\lambda = \frac{|\lambda|!}{k_1 \cdots k_{|\lambda|}}.$$

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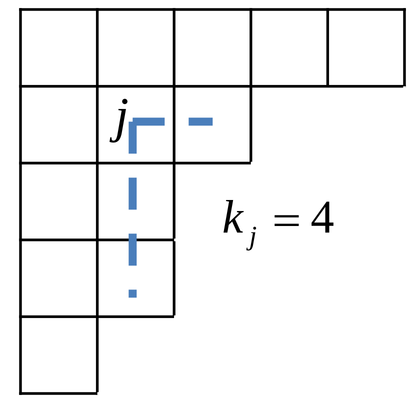

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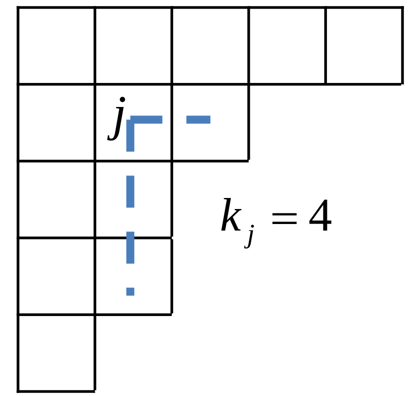


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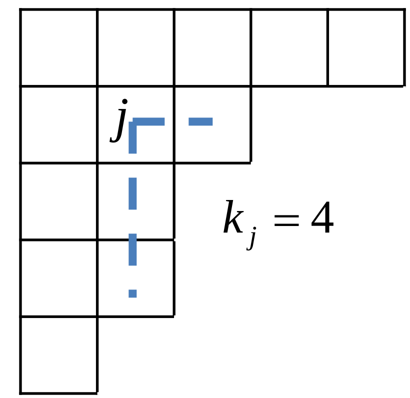


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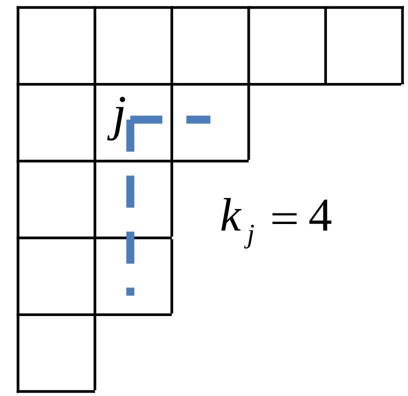
Theorem 5: We have $W[\lambda, \overline{f}\,] = S_\lambda(h)\; W[\overline{f}\,]$ [Giambelli's formula],

<u>Theorem 4</u> (Derivatives of the Wronskian in terms of the generalized Wronskians):

$$W[\bar{f}]^{(k)}(t) = \sum\nolimits_{|\lambda|=k} c_\lambda W[\lambda, \bar{f}](t), \quad \text{where} \quad c_\lambda = \frac{|\lambda|!}{k_1 \cdots k_{|\lambda|}}.$$

*(hook formula)*

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$$h_k \ W[\lambda, \bar{f}] = \sum\nolimits_\mu W[\mu, \bar{f}] \quad \text{[Pieri's formula],}$$

the sum over all partitions $\mu \in P_r$ satisfying

$$|\mu| = k + |\lambda|, \ \ \mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \ldots \mu_{r-1} \geq \lambda_{r-1}.$$

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Mapping $h_j \mapsto \sigma_{(j)}$ defines a surjection $B_r \to H^*(G, \mathbb{Z})$, and its kernel gives relations on generators ( $B_r = \mathbb{Q}[e_1, \dots, e_r]$ the ring of symmetric functions).

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$$\sigma_\lambda = S_\lambda(\overline{\sigma}), \ \sigma_{(k)}\sigma_\lambda = \sum\nolimits_\mu \sigma_\mu,$$

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<u>Corollary</u>: The $\mathbb{Z}$-module isomorphism $\mathcal{W}_r \to H_*(G, \mathbb{Z})$ defined by $W[\lambda, \overline{f}\,] \mapsto \Omega_\lambda$ is an isomorphism of $H^*(G, \mathbb{Z})$ -modules.

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Corollary: The correspondence $\overline{u}_\lambda \leftrightarrow W[\lambda, \overline{u}]$ gives the $B_r$-module isomorphism $\mathcal{W}_r \sim \wedge^r K_r$ .

We call

$$B_r = \mathbb{Q}[e_1, \ldots, e_r]$$ the *r-th Bosonic space;*

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If $r = \infty$ we have $B_\infty = \mathbb{Q}[e_1, e_2, \dots]$, $K_\infty = Span_{B_\infty}\{u_0, u_{-1}, \dots\}$;

$$u_j(t) = \sum_{n \geq 0} h_{n+j} \frac{t^n}{n!}\ ,\quad j \in \mathbb{Z},$$ all are solutions to the *"ODE of infinite order"*,

where $\{h_i\}_{i \in \mathbb{Z}}$ are defined in the same way as for $r < \infty$,

$$E_\infty(t) \sum\nolimits_{n \in \mathbb{Z}} h_n t^n = 1,\ E_\infty(t) = 1 + \sum\nolimits_{j \geq 1} (-1)^j e_j t^j.$$

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Denote $\{x_j\}_{j\geq 1}$ the elements of $B_r$ defined by the generating function

$$\sum_{j\geq 1} x_j z^j = -\log(1 - e_1 z + e_2 z^2 - \ldots + (-1)^r e_r z^r).$$

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$$\boxed{\text{If } r = \infty \text{, then } \sum_{n\geq 1} x_n t^n = -\log E_\infty(t),\ E_\infty(t) = 1 + \sum_{j\geq 1} (-1)^j e_j t^j.}$$

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They both can be naturally extended first to operators in $\wedge^r K_r$, and then in $F_0^r$.

We denote them $\hat{\partial}_i$ and $\hat{D}^k$, resp.

(recall: $K_r = Span_{B_r}\{u_0, u_1, \ldots, u_{1-r}\} = Span_{\mathbb{Q}}\{u_j, \;\; j \geq -r+1\}$).

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Denote $\mathscr{H}_r$ the Lie algebra over $\mathbb{Q}$ generated by $\{p_i, \hbar\}_{-r \le i \le r}$ such that

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$\mathscr{H}_\infty$ is the *oscillator Heisenberg Algebra* generated over $\mathbb{C}$ by $\{p_i, \hbar\}_{i \in \mathbb{Z}}$ such that

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On this way, finite-dimensional counterparts of vertex operators appear

[L. Gatto and P. Salehyan, arXiv 13.10.5132]