Maps that take lines to plane curves

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Planarizations

Definition

A planarization is a sufficiently smooth mapping $f : U \subset \mathbb{RP}^2 \to \mathbb{RP}^3$ such that, for every line $L \subset \mathbb{RP}^2$, the set $f(U \cap L)$ is planar.

Definition

Two planarizations $f : U \to \mathbb{RP}^3$ and $g : V \to \mathbb{RP}^3$ are equivalent if there is a nonempty open subset $W \subset U \cap V$ such that $f = g$ on W, up to projective transformations of the source and target spaces.

Problem

Classify planarizations according to this equivalence relation.

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The Fundamental Theorem of Projective Geometry

Theorem (Möbius, 1827)

Suppose that $f : \mathbb{RP}^n \to \mathbb{RP}^n$ is a continuous one-to-one map taking all straight lines to straight lines. Then f is a projective transformation, i.e., a projectivization of a linear isomorphism $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.

Theorem (von Staudt)

The continuity assumption is superfluous.

Remark

This theorem has local versions.

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Classical geometers

August Möbius
1790–1868

Karl Georg Christian von Staudt
1798–1867

Motivation

- An extension of the Fundamental Theorem of Projective Geometry
- Let $\mathcal{L}$ be a linear system of curves (e.g., the family of all lines, circles, conics, etc.). Studying mappings $f : U \subset \mathbb{RP}^2 \to \mathbb{RP}^2$ taking line segments to curves from $\mathcal{L}$ is related with studying planarizations.

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Trivial cases

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A planarization $f : U \to \mathbb{RP}^3$ is trivial if $f(U)$ lies in a plane.

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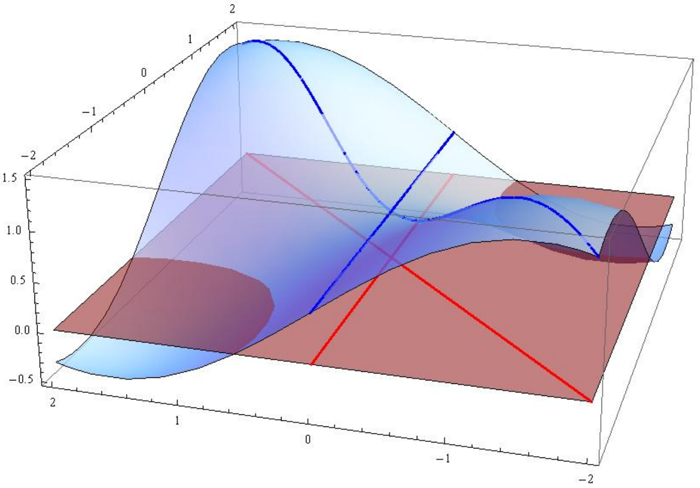
Co-trivial planarizations



Non-trivial examples

Definition

A quadratic rational mapping is a rational mapping $f : \mathbb{RP}^2 \dashrightarrow \mathbb{RP}^2$ given in homogeneous coordinates by homogeneous polynomials of degree 2:

$$f[x_0 : x_1 : x_2] = [y_0 : y_1 : y_2 : y_3], \quad y_\alpha = \sum_{i,j=0}^{2} a_\alpha^{i,j} x_i x_j.$$

Example

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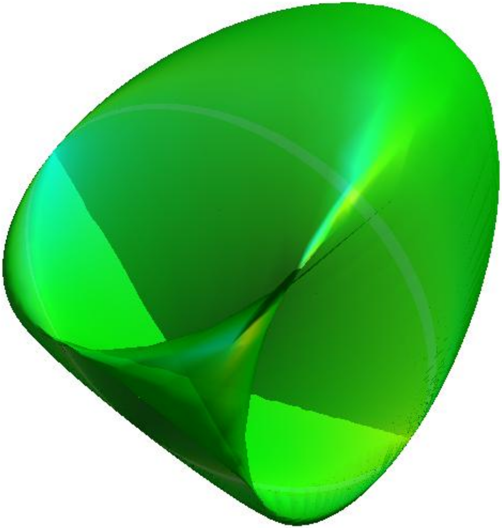

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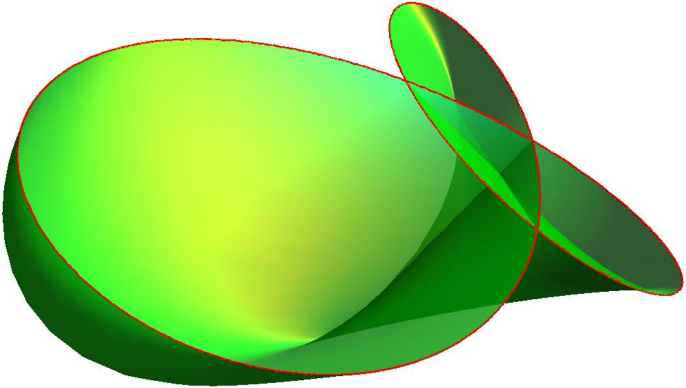

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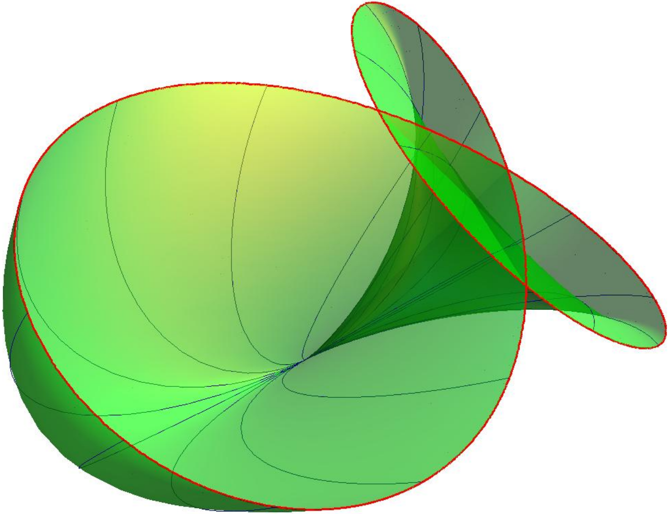

Duality

- This is an implementation of projective duality for planarizations.
- For every planarization $f : U \to \mathbb{RP}^3$, there is the dual planarization $f^* : U^* \to \mathbb{RP}^{3*}$.
- The open set U^*, possibly empty, is defined as the set of all lines $L \in \mathbb{RP}^{2*}$ such that $f(L \cap U)$ lies in a unique plane P_L.
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The result

Theorem

Every planarization $f : U \to \mathbb{R}\mathrm{P}^3$ is equivalent to a planarization that is

- *trivial, OR*
- *co-trivial, OR*
- *quadratic, OR*
- *dual quadratic.*

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The classification

Theorem

There are 16 equivalence classes of non-(co)-trivial planarizations:

(Q_1): $[x:y:z] \mapsto [xy:xz:yz:x^2+y^2+z^2]$

(Q_2): $[x:y:z] \mapsto [xy:xz:yz:x^2-y^2+z^2]$

(Q_3): $[x:y:z] \mapsto [x^2+y^2:y^2+z^2:xz:yz]$

(Q_4): $[x:y:z] \mapsto [x^2-y^2:xy:yz:z^2]$

(Q_5): $[x:y:z] \mapsto [xz-yz:x^2:y^2:z^2]$

(Q_6): $[x:y:z] \mapsto [x^2:xz-y^2:yz:z^2]$

(Q_7): $[x:y:z] \mapsto [y^2-z^2:xy:xz:yz]$

(Q_8): $[x:y:z] \mapsto [xy:xz:y^2:z^2]$

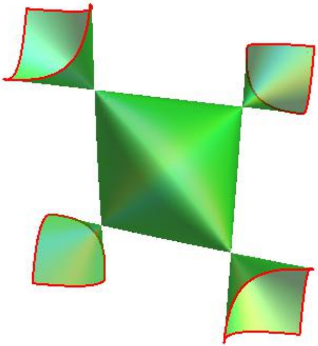
(Q_9): $[x:y:z] \mapsto [xy:xz-y^2:yz:z^2]$

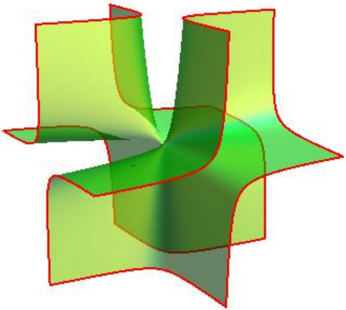
(Q_{10}): $[x:y:z] \mapsto [x^2:xy:y^2:z^2]$ ...

The classification

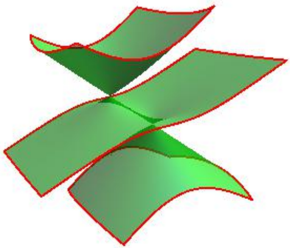
(C_1): $[x:y:z] \mapsto [z(x^2+y^2) : y(x^2+z^2) : x(y^2+z^2) : xyz]$
(C_2): $[x:y:z] \mapsto [z(x^2-y^2) : y(x^2+z^2) : x(y^2-z^2) : xyz]$
(C_3): $[x:y:z] \mapsto [x^2z : z(x^2+y^2) : x(x^2+y^2-z^2) : y(x^2+y^2+z^2)]$
(C_4): $[x:y:z] \mapsto [x^2y : x(x^2-y^2) : z(x^2+y^2) : yz^2]$
(C_5): $[x:y:z] \mapsto [x^2(x+y) : y^2(x+y) : z^2(x-y) : xyz]$
(C_6): $[x:y:z] \mapsto [x^3 : xy^2 : 2xyz-y^3 : z(xz-y^2)]$.

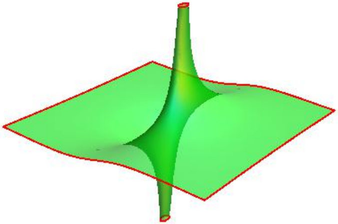
Planarizations (C_1) and (C_2)





Planarizations (C_3) and (C_4)





Planarizations (C_5) and (C_6)