Fractional Dynamic Oligopoly Model

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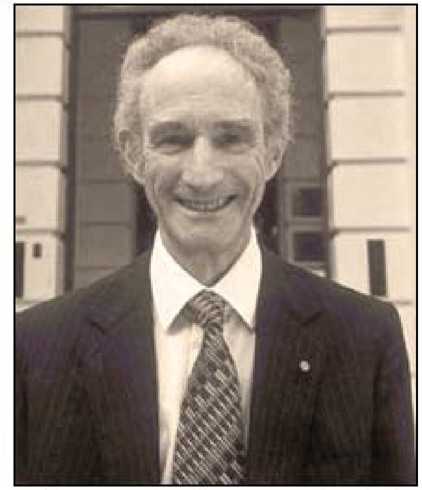
Topics

- Logistics Model
- Fractional Calculus
- Complex Oligopoly
- Cournot duopoly model
- Best Response Dynamics
- Reaction function
- Adjustment process
- Fractional Discrete Map
- Numerical Results
- Conclusion

Logistic Equation

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

where N the population with respect to time
r is rate of maximum population growth
K is carrying capacity



Professor Lord Robert May of Oxford, AC, OM, Kt

Nature Vol. 261 June 10 1976 459

review article

Simple mathematical models with very complicated dynamics

Robert M. May*

First-order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. This is an interpretive review of them.

Logistic Model

The logistic model is explained the constant population growth rate which not includes the limitation on food supply or spread of diseases The solution curve of the model is increase exponentially from the multiplication factor up to saturation limit which is maximum carrying capacity.

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

where N the population with respect to time

r is rate of maximum population growth

K is carrying capacity

Logistic model (Continuous time)

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

Note that if $N > K$, then $dN/dt < 0$ so the population declines.

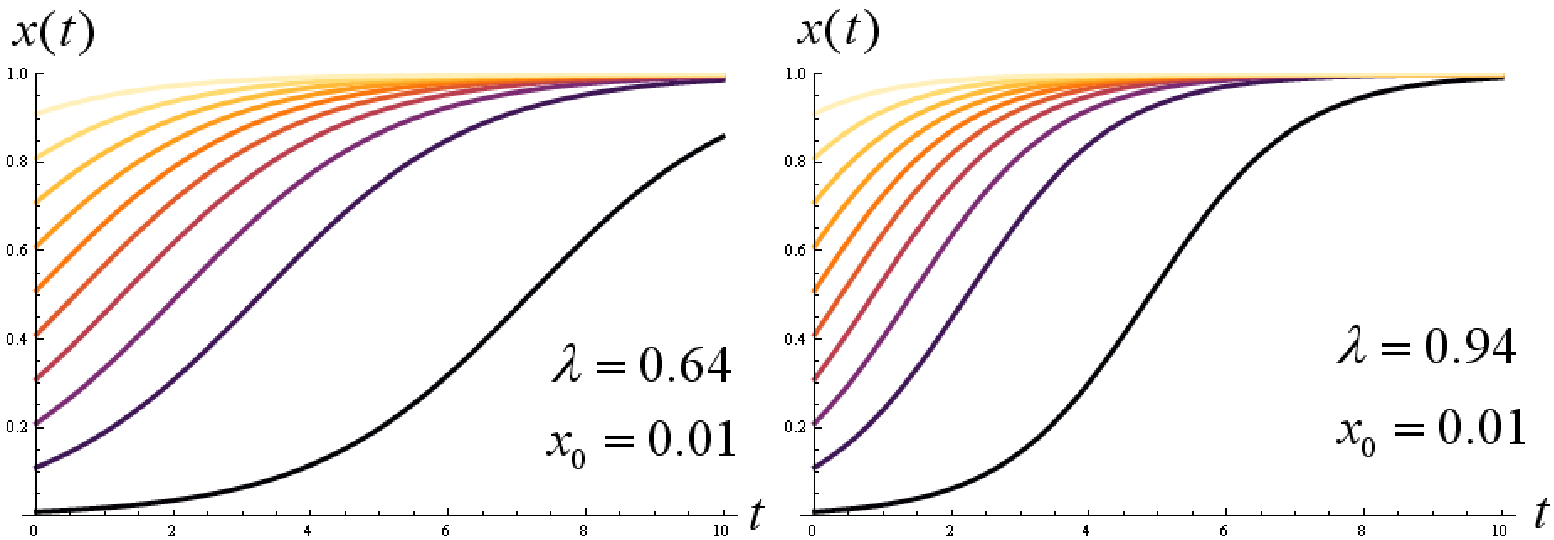
The equilibrium solutions of the logistic equation is when the solutions is constant or $dN/dt = 0$, then $0 = r(1 - N/K)$, the two solutions are $N(t) = 0$ and $N(t) = K$.

The differential equation is solved by integrating.

The general solution is in the form as following.

$$N(t) = \frac{K}{1 + Ae^{-rt}} \quad \text{where} \quad A = \frac{K - x(0)}{x(0)}$$

Plot of Logistic curve



Discrete Logistic function

The nonlinear model is usually cast in a non dimensional form as following equation

$$x_{n+1} = \lambda x_n (1 - x_n)$$

This equation has at least one equilibrium point, $x = 0$, for $\lambda > 1$, two equilibrium points exist (i.e., solutions of the equation $x = \lambda x(1-x)$. To determine the stability of a map $x_{n+1} = f(x)$, one looks at the value of the slope $|f'(x)|$ evaluated at the fixed point.

The fixed point is unstable if $|f'(x)| > 1$. In the case of the logistic equation when $1 < \lambda < 3$, there are two fixed points, namely, $x = 0$ and $x = (\lambda - 1)/\lambda$; the origin is unstable and the other point is stable.

Fractional Calculus

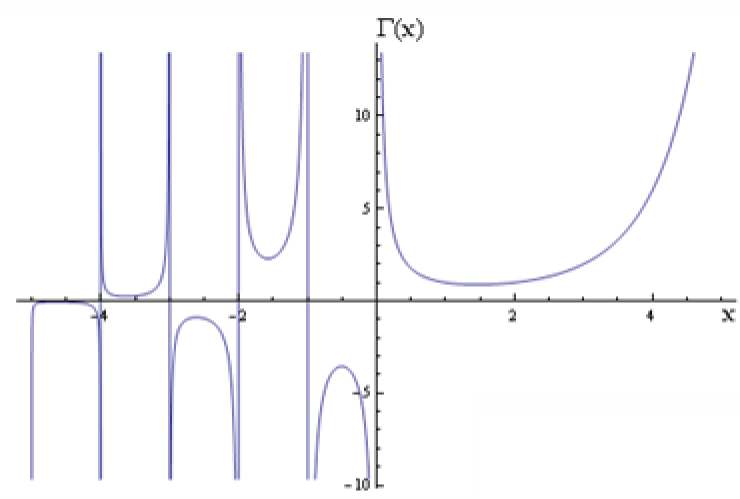
- The subject of fractional calculus is calculus of integrals and derivatives of any arbitrary real or complex order
- 300 year old mathematical topics
- The concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by L'Hopital to Leibniz, which sought the meaning for the derivative of order n when $n=1/2$.
- Recently have applications in many fields

Definition of Gamma function

Definition $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}$$

Gamma Function	
$\Gamma(\frac{3}{2})$	$\frac{\sqrt{\pi}}{2}$
$\Gamma(\frac{5}{2})$	$\frac{3\sqrt{\pi}}{4}$
$\Gamma(\frac{7}{2})$	$\frac{15\sqrt{\pi}}{8}$
$\Gamma(\frac{9}{2})$	$\frac{105\sqrt{\pi}}{16}$
$\Gamma(\frac{11}{2})$	$\frac{945\sqrt{\pi}}{32}$



Grunwald-Letnikov Definition

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}$$

$$\begin{aligned} f''(t) = \frac{d^2 f}{dt^2} &= \lim_{h\to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \left\{ \frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h} \right\} \\ &= \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2} \end{aligned}$$

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h\to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3}$$

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh).$$

Example of Fractional derivative

$$f(x) = x^k$$

$$\frac{d}{dx} f(x) = kx^{k-1}$$

$$\frac{d^a}{dx^a} x^k = \frac{k!}{(k-a)!} x^{k-a}$$

$$\frac{d^a}{dx^a} x^k = \frac{\Gamma(k+1)}{\Gamma(k-a+1)} x^{k-a}$$

$$\frac{d^{1/2}}{dx^{1/2}} x = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} = 2\pi^{-\frac{1}{2}} x^{\frac{1}{2}} = \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}}$$

$$\frac{d^{1/2}}{dx^{1/2}} 2\pi^{-\frac{1}{2}} x^{\frac{1}{2}} = 2\pi^{-\frac{1}{2}} \frac{\Gamma(1+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)} x^{\frac{1}{2}-\frac{1}{2}} = 2\pi^{-\frac{1}{2}} \frac{\Gamma(\frac{3}{2})}{\Gamma(1)} x^0 = \frac{1}{\Gamma(1)} = 1$$

Example (cont)

$$f(x) = x^a$$

$$D^1 x^a = a x^{a-1}$$

$$D^n x^a = x^{a-n} \prod_{m=0}^{n-1} (a-m) = \frac{a!}{(a-n)!} x^{a-n}$$

$$D^\alpha x^a = \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} x^{a-\alpha}$$

Fractional-order Logistic

$$f(x) = \lambda x(1-x)$$

$$D_x^{\gamma} f(x) = \lambda\left[\frac{\Gamma(2)x^{1-n}}{\Gamma(2-n)} - \frac{\Gamma(3)x^{2-n}}{\Gamma(3-n)}\right]$$

$$= \lambda\left[\frac{x^{1+\gamma}}{\Gamma(2+\gamma)} - \frac{2x^{2+\gamma}}{\Gamma(3+\gamma)}\right]$$

$$= \lambda\left[\frac{x^{1+\gamma}}{(1+\gamma)\Gamma(1+\gamma)} - \frac{2x^{2+\gamma}}{(2+\gamma)(1+\gamma)\Gamma(1+\gamma)}\right]$$

$$= \frac{\lambda x^{1+\gamma}}{(1+\gamma)\Gamma(1+\gamma)}\left[1 - \frac{2x}{2+\gamma}\right]$$

$$= \frac{\lambda x^{1+\gamma}}{\Gamma(\gamma+2)}\left[1 - \frac{2x}{2+\gamma}\right]$$

Complex oligopoly dynamics

- Original oligopoly model is in the form of linear function
- Recent studies of oligopoly model are in the forms of nonlinear function
- Nonlinear dynamical system many type of complexity arises include limit cycle, periodic doubling, strange attractor

Cournot Duopoly model

- Bischi et al., have introduced the discrete-time Cournot duopoly model with partial adjustment to the Best Response where the two firms in the model produce the same product.
- The equilibrium is described the strategy sets from which each firms choose their output quantities and the resulting instantaneous payoffs.

Cournot Duopoly model (cont)

- The Cournot reaction function are described by the quantities that their rival firms produced

$$\max_{q_1} \Pi_1(q_1, q_2) = \Pi_1(r_1(q_2), q_2)$$

$$\max_{q_2} \Pi_2(q_1, q_2) = \Pi_2(q_1, r_2(q_1))$$

Cournot tatonnement

- The Cournot tatonnement or Best Response Dynamic can now be described in terms of the reactions functions as

$$q_1(t+1) = r_1(q_2(t))$$
$$q_2(t+1) = r_2(q_1(t))$$

- where r_1 and r_2 are often referred to as Best Replies (or reaction functions).

Reaction function Bischi & Kopel type.

- Stackelberg leadership in which one firm becomes the leader by taking the reaction functions of the other firms
- Bischi and Kopel propose the reaction function in term of couple logistic function

Reaction Functions

- Each player sets the current output equal to the best response (i.e., current period pay-off maximizing choice) to the last period output of its opponent.

$$r_1(q_2) = \mu_1 q_2 (1 - q_2)$$
$$r_2(q_1) = \mu_2 q_1 (1 - q_1)$$

Adjustment process

- Bischi et al., have proposed the adjustment process in case that the firms are insecure if the forecasts of the opponent’s behavior is correct or if the decision making process involves some other thing then the firms may use an adjustment process which is described by

$$q_1(t+1) = q_1(t) + \lambda_1(r_1(q_2(t)) - q_1(t))$$
$$q_2(t+1) = q_2(t) + \lambda_2(r_2(q_1(t)) - q_2(t))$$

Generalize logistic function

- The generalize fractional-order logistic model is the variation form of logistic model by fractional differential operator

$$D_x^{\gamma} x = \frac{\mu x^{1+\gamma}}{\Gamma(\gamma+2)}(1-\frac{2x}{2+\gamma})$$

where γ is fractional order and

$\Gamma(\cdot)$ is Euler Gamma function

Fractional Discrete map

- The fractional-order logistic map obtains by replace the fractional differential equation with fractional discrete map as follow

$$x(t+1)=\frac{\mu x^{1+\gamma}(t)}{\Gamma(q+2)}(1-\frac{2x(t)}{2+\gamma})$$

where γ is the order in differential operator.

Fractional Best Response Model

- By substitute generalize logistic function in Best Response Dynamic equation

$$q_1(t+1) = q_1(t) + \lambda_1(r_1(q_2(t)) - q_1(t))$$
$$q_2(t+1) = q_2(t) + \lambda_2(r_2(q_1(t)) - q_2(t))$$

- Substitute the logistic function in reaction function, then we have a iterative map T

$$T:\begin{cases} q_1' = (1-\lambda_1)q_1 + \lambda_1\mu_1 q_2(1-q_2) \\ q_2' = (1-\lambda_2)q_2 + \lambda_2\mu_2 q_1(1-q_1) \end{cases}$$

Fractional Best Response Model

- By substitute the iterative map T with generalize logistic function then we obtained the new form of map

$$T:\begin{cases} q_1(t+1)=(1-\lambda_1)q_1(t)+\dfrac{\lambda_1\mu_1 q_2^{1+\gamma}}{\Gamma(\gamma+2)}(1-\dfrac{2q_2}{2+\gamma}) \\ q_2(t+1)=(1-\lambda_2)q_2(t)+\dfrac{\lambda_2\mu_2 q_1^{1+\gamma}}{\Gamma(\gamma+2)}(1-\dfrac{2q_1}{2+\gamma}) \end{cases}$$

Example Fractional Map

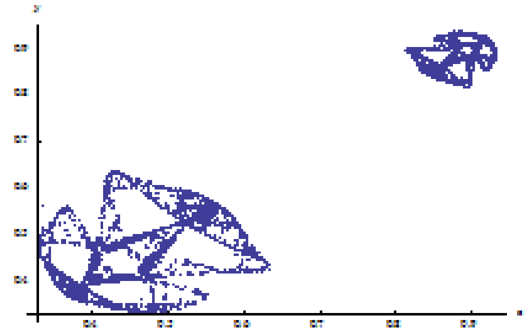
- An example of fractional map for $\gamma = 1/100$

$$T:\begin{cases} q_1(t+1) = (1-\lambda_1)q_1(t) + \dfrac{\lambda_1\mu_1 q_2^{101/100}}{\Gamma(201/100)}(1-\dfrac{200q_2}{201}) \\ q_2(t+1) = (1-\lambda_2)q_2(t) + \dfrac{\lambda_2\mu_2 q_1^{101/100}}{\Gamma(201/100)}(1-\dfrac{200q_1}{201}) \end{cases}$$

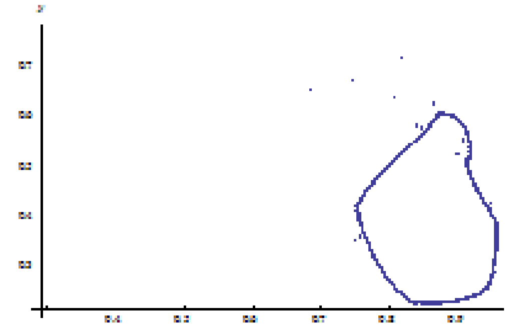
Numerical results

- Present in the graphic form of phase space at different parameters sets
- There are different types of dynamical behaviour arise in duopoly model include periodic, limit cycle, and chaotic

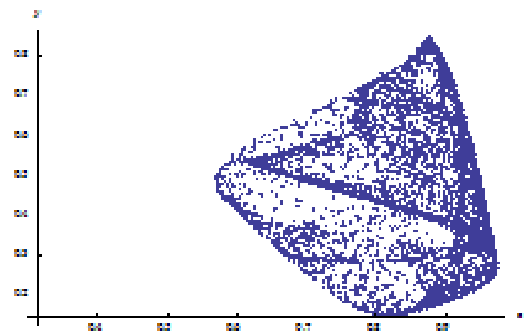
Phase space of integer order



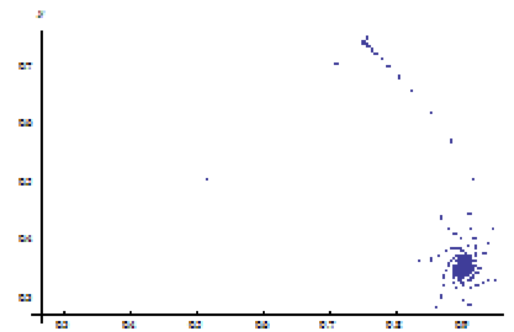
$\lambda = 0.871, \mu = 3.93$



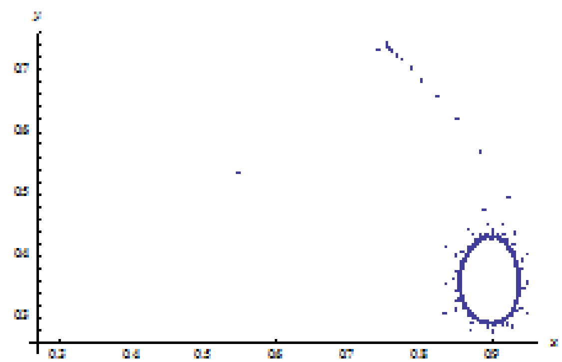
$\lambda = 0.595, \mu = 3.93$



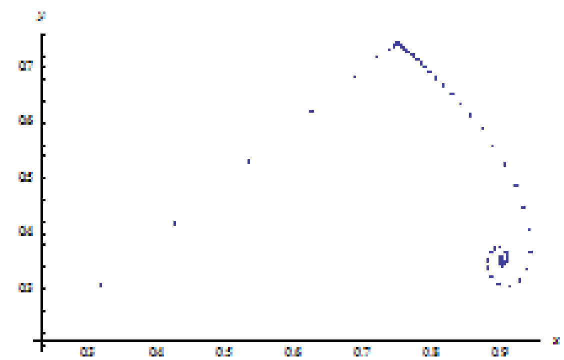
$\lambda = 0.74, \mu = 3.93$



$\lambda = 0.419, \mu = 3.93$

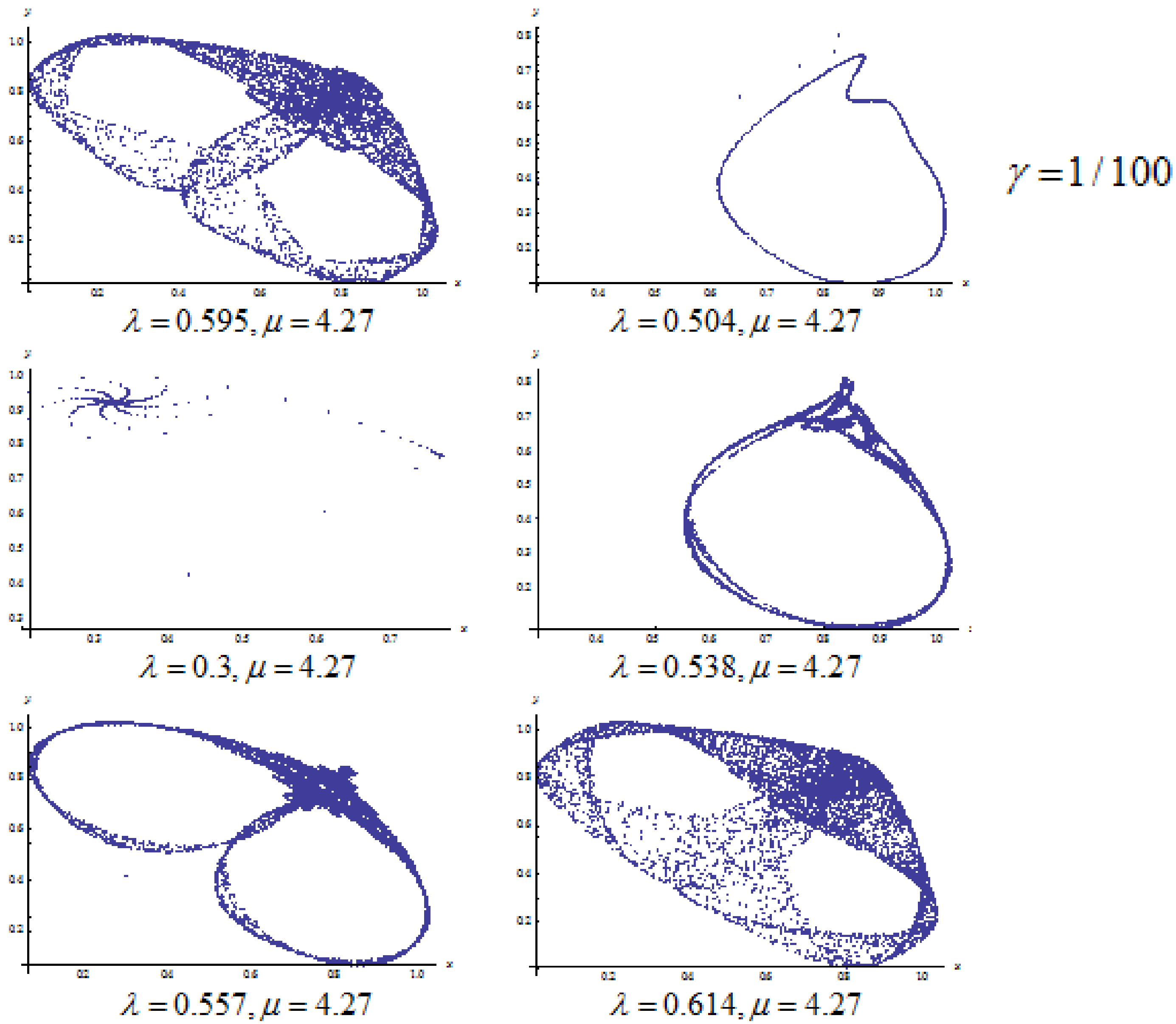


$\lambda = 0.452, \mu = 3.93$



$\lambda = 0.204, \mu = 3.93$

Phase space of fractional order



Conclusion

- We have study the dynamic of duopoly model where the reaction function described by logistic function further with fractional calculus.
- Two-dimensional duopoly phase space have present with different parameters.
- The numerical results are aimed to study behavior of duopoly at fractional-order compare to the integer-order.
- The graphical results of phase portrait are provided the behavior of the model range from limit-cycle, Hopf-type bifurcation and strange attractor.
- Theory of fractional calculus is feasible to examine different behavior at difference regime.