Higher dimensional Ellentuck spaces

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References for this talk

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For a survey of pre-2014 Tukey and related Ramsey space results, see

[Dobrinen] *Survey on the Tukey theory of ultrafilters*, Zbornik Radova, Mathematical Institute of the Serbian Academy of Sciences, 2015.

A very brief review of Tukey reduction between ultrafilters

Def. $\mathcal{V}$ is *Tukey reducible* to $\mathcal{U}$ ($\mathcal{V} \le_T \mathcal{U}$) if there is a map $f : \mathcal{U} \to \mathcal{V}$ such that each f-image of a filter base for $\mathcal{U}$ is a filter base for $\mathcal{V}$.

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The Tukey equivalence class of an ultrafilter is called its *Tukey type*.

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For more overview, see my recent survey paper.

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[Milovich 2008 (initial work on Tukey and Isbell's Problem)]

[Dobrinen/Todorcevic 2011 (embeddings), 2014 and 2015 (initial structures)]

[Dobrinen Continuous cofinal maps 2010 preprint - (extended to become Continuous and other canonical cofinal maps (2015))]

[Raghavan/Todorcevic 2012 (RK versus Tukey and first initial structure result for Ramsey ultrafilters)]

[Dobrinen/Mijares/Trujillo submitted 2014 (Boolean algebras as initial structures for Tukey and a rich collection of initial structures for RK)]

[Raghavan/Shelah submitted 2014 (embedding $\mathcal{P}(\omega)/\text{fin}$ into RK and Tukey types of p-points)]

The Forcing $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$

$\mathrm{Fin} \otimes \mathrm{Fin} = \{X \subseteq \omega \times \omega : \forall^\infty i \in \omega\ \{j \in \omega : (i,j) \in X\} \text{ is finite}\}$.

That is, for all but finitely many i, the i-th fiber of X is finite.

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$\mathcal{G}_2$ is neither a p-point, nor a Fubini product of p-points, but the projection to the first coordinates $\pi_1(\mathcal{G}_2)$ is a Ramsey ultrafilter.

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This left open what exactly *is* Tukey reducible to $\mathcal{G}_2$; i.e. What is the *initial Tukey structure below* $\mathcal{G}_2$.

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We will thin this even more and put more restrictions on the subsets of $\omega \times \omega$ we allow in order to obtain a topological Ramsey space $\mathcal{E}_2$ which is forcing equivalent to $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$. Our space $\mathcal{E}_2$ looks and acts *like* ω copies of the Ellentuck space, given a judiciously chosen finitization map.

Review

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Def. $\mathcal{X} \subseteq [\omega]^\omega$ is *Ramsey* iff for each $[s, X]$, there is $s \sqsubset Y \subseteq X$ such that either $[s, Y] \subseteq \mathcal{X}$ or $[s, Y] \cap \mathcal{X} = \emptyset$.

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Associated Ultrafilter: Ramsey ultrafilter forced by $([\omega]^\omega, \le^*)$, has 'complete combinatorics'.

Topological Ramsey spaces $(\mathcal{R}, \leq, r)$

Basic open sets: $[a, A] = \{X \in \mathcal{R} : \exists n(r_n(X) = a) \text{ and } X \leq A\}$.

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Def. [Todorcevic] A triple $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* if every subset of $\mathcal{R}$ with the Baire property is Ramsey, and if every meager subset of $\mathcal{R}$ is Ramsey null.

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Abstract Ellentuck Theorem. [Todorcevic]
If $(\mathcal{R}, \leq, r)$ satisfies **A.1 - A.4** and $\mathcal{R}$ is closed (in $\mathcal{AR}^{\mathbb{N}}$), then $(\mathcal{R}, \leq, r)$ is a topological Ramsey space.

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n-th Apppproximations: $\mathcal{AR}_n = \{r_n(X) : X \in \mathcal{R}\}$.
Finite Approximations: $\mathcal{AR} = \bigcup_{n<\omega} \mathcal{AR}_n$.

tRs's force ultrafilters with complete combinatorics

Thm. [DiPrisco/Mijares/Nieto (submitted 2014)] Let $\mathcal{R}$ be a topological Ramsey space. If there exists a supercompact cardinal, then every selective coideal $\mathcal{U} \subseteq \mathcal{R}$ is $(\mathcal{R}, \leq^*)$-generic over $L(\mathbb{R})$.

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The upshot is that if we show that $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$ is forcing equivalent to some topological Ramsey space, then (with minor modifcations to their proofs) the above theorem implies that the generic ultrafilter $\mathcal{G}_2$ has 'complete combinatorics'.

The structure behind $\mathcal{E}_2$: $(\omega^{\not\downarrow \leq 2}, \prec)$

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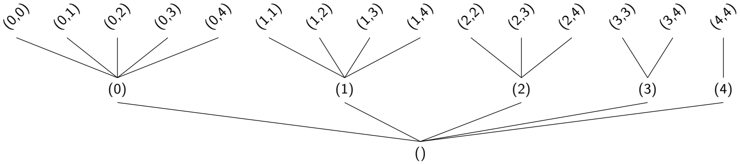
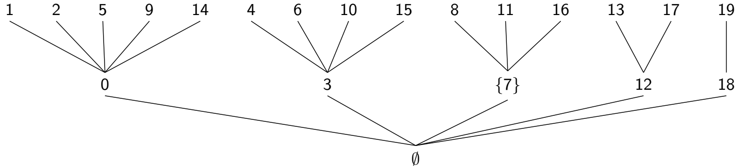


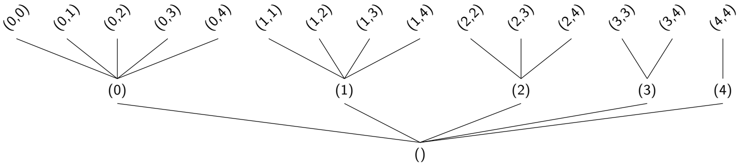
Figure: $\omega^{\not\downarrow \leq 2}$

The well-order $(\omega^{\not\downarrow \leq 2}, \prec)$ begins as follows:

$() \prec (0) \prec (0,0) \prec (0,1) \prec (1) \prec (1,1) \prec (0,2) \prec (1,2) \prec (2) \prec (2,2) \prec$

Constructing the maximal member of $\mathcal{E}_2$





$$\begin{array}{cccccccccc} \emptyset & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ () \prec & (0) \prec & (0,0) \prec & (0,1) \prec & (1) \prec & (1,1) \prec & (0,2) \prec & (1,2) \prec & (2) \prec & (2,2) \prec \end{array}$$

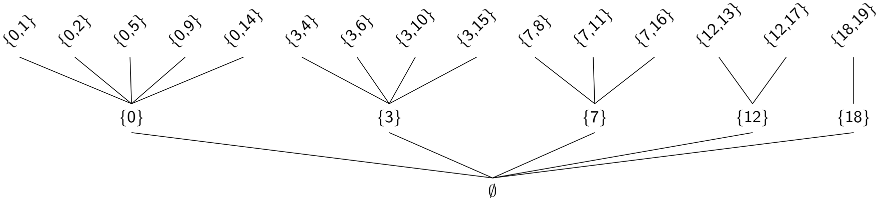
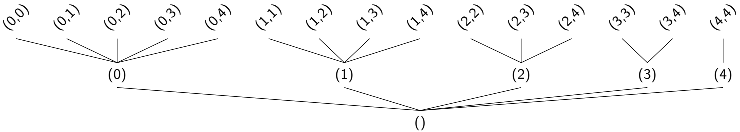


Figure: Maximum element $\mathbb{W}_2 \subseteq [\omega]^2$ of $\mathcal{E}_2$



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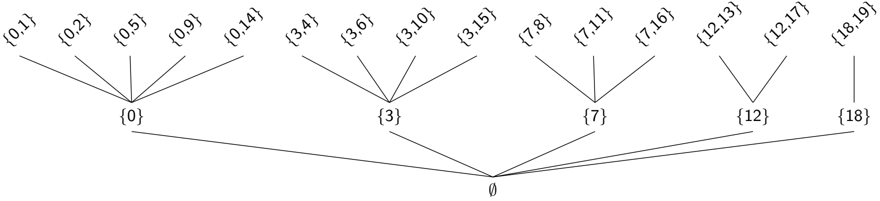

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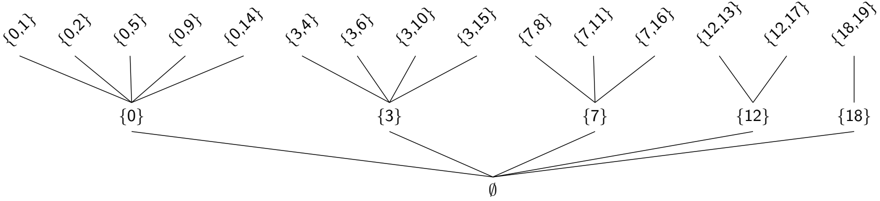

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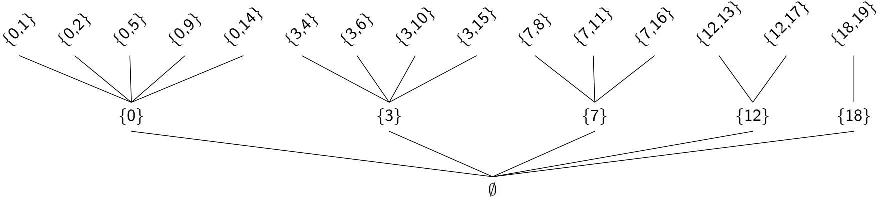

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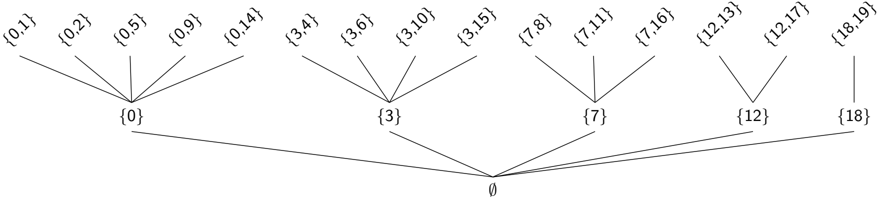

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Typical finite approximations to members of $\mathcal{E}_2$

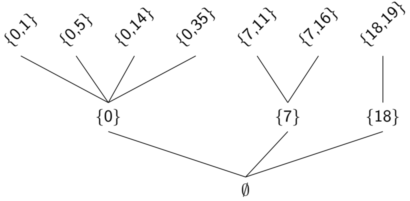

Figure: $r_7(X)$

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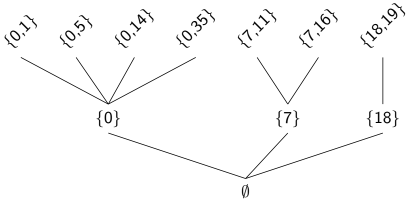

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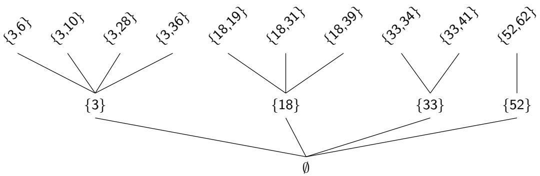

Figure: $r_{10}(Y)$

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In order to satisfy the Amalgamation Axiom (**A3 (2)**) in Todorcevic's characteriztion of topological Ramsey spaces, some such requirement is necessary.

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Def. A set $\mathcal{X} \subseteq \mathcal{E}_2$ is *Ramsey* iff for each basic open $[a, X]$, there is a $Y \in [a, X]$ such that either $[a, Y] \subseteq \mathcal{X}$ or $[a, Y] \cap \mathcal{X} = \emptyset$.

$\mathcal{AR}$ denotes the collection of all finite approximations of members of $\mathcal{E}_2$.

For $a \in \mathcal{AR}$ and $X \in \mathcal{E}_2$, $[a, X] := \{Y \in \mathcal{E}_2 : a \sqsubset Y \subseteq X\}$.

The Ellentuck topology is generated by basic open sets of the form $[a, X]$, where $a \in \mathcal{AR}$ and $X \in \mathcal{E}_2$.

$\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$ is forcing equivalent to a new topological Ramsey space

$(\mathcal{E}_2, \subseteq^{\mathrm{Fin}^{\otimes 2}})$ is forcing equivalent to $((\mathrm{Fin}^{\otimes 2})^+, \subseteq^{\mathrm{Fin}^{\otimes 2}})$.

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(Below any member $A \in (\mathrm{Fin}^{\otimes 2})^+$ is some $B \subseteq A$ which is an isomorphic copy of $\mathbb{W}_2$, and below B, there is a dense subset of $(\mathrm{Fin}^{\otimes 2})^+ \restriction B$ isomorphic to $\mathcal{E}_2$.)

Higher order forcings

$\text{Fin}^{\otimes 3}$ is the ideal on $\omega \times \omega \times \omega$ such that

$X \subseteq \omega^3$ is in $\text{Fin}^{\otimes 3}$ iff for all but finitely many $i < \omega$, the i-th fiber of X, $\{(j,k) \in \omega \times \omega : (i,j,k) \in X\}$, is in $\text{Fin} \otimes \text{Fin}$.

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$\mathcal{P}(\omega^3)/\text{Fin}^{\otimes 3}$ adds a generic ultrafilter $\mathcal{G}_3$ on ω^3 such that its projection to the first two coordinates is a generic ultrafilter forced by $\mathcal{P}(\omega^2)/\text{Fin}^{\otimes 2}$, and its projection to the first coordinate is a Ramsey ultrafilter forced by $\mathcal{P}(\omega)/\text{Fin}$.

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We thin $(\mathrm{Fin}^{\otimes 3})^+$ to a topological Ramsey space $\mathcal{E}_3$ forcing equivalent (when partially ordered by $\subseteq^{\mathrm{Fin}^{\otimes 3}}$) to $\mathcal{P}(\omega^3)/\mathrm{Fin}^{\otimes 3}$.

The structure behind $\mathcal{E}_3$

The well-order $(\omega^{\not\downarrow\leq 3}, \prec)$ begins as follows:

$$\begin{aligned}\emptyset &\prec (0) \prec (0,0) \prec (0,0,0) \prec (0,0,1) \prec (0,1) \prec (0,1,1) \prec (1) \prec (1,1) \\ &\prec (1,1,1) \prec (0,0,2) \prec (0,1,2) \prec (0,2) \prec (0,2,2) \prec (1,1,2) \\ &\prec (1,2) \prec (1,2,2) \prec (2) \prec (2,2) \prec (2,2,2) \prec (0,0,3) \prec \cdots\end{aligned} \tag{1}$$

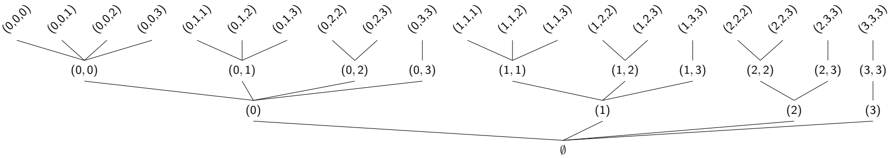


Figure: $\omega^{\not\downarrow\leq 3}$

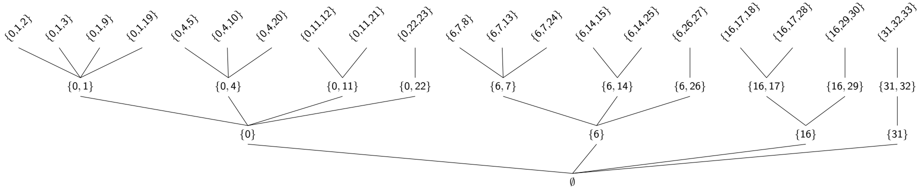
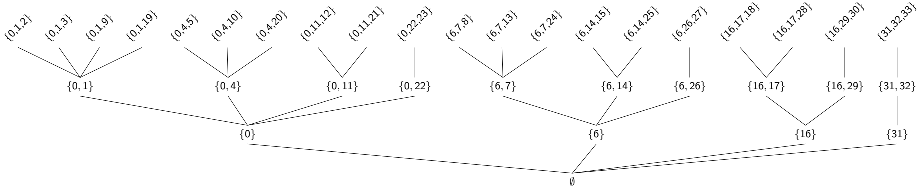


Figure: The maximum member of $\mathcal{E}_3$, $\mathbb{W}_3 \subseteq [\omega]^3$



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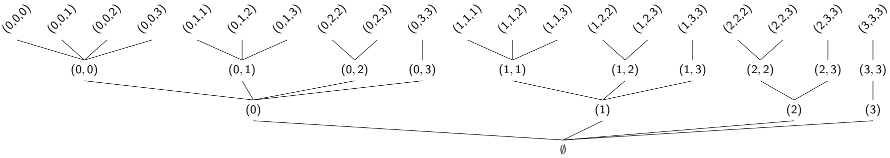

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The space $\mathcal{E}_3$

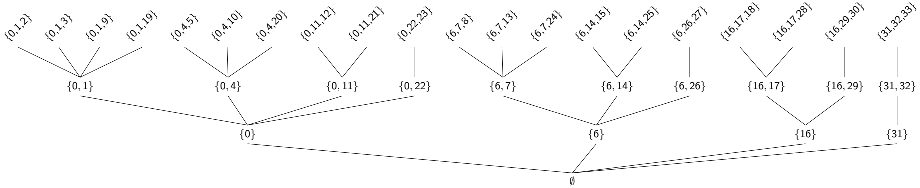

Figure: $\mathbb{W}_3$

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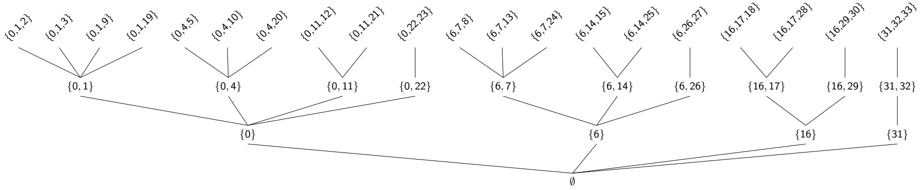

Figure: $\mathbb{W}_3$

$X \in \mathcal{E}_3$ iff $X \subseteq \mathbb{W}_3$ and $X \cong \mathbb{W}_3$ as a tree, and also with respect to the $\prec$ order of the node labels.

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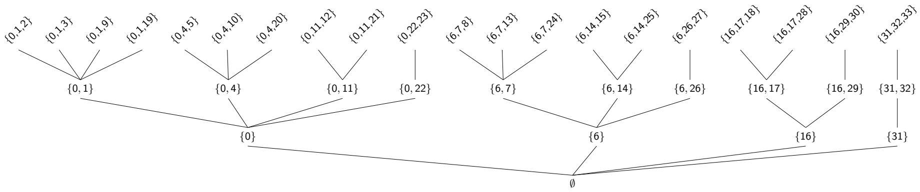

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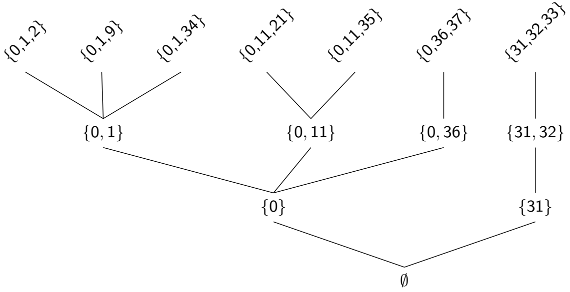
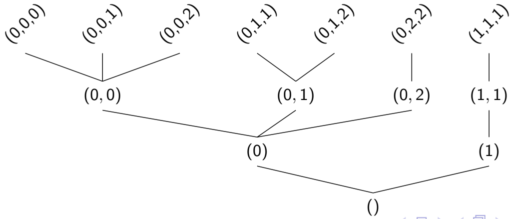


Figure: $r_7(Y)$, a typical finite approximation to a member of $\mathcal{E}_3$



We now define the spaces $\mathcal{E}_k$, $k \geq 2$, in general.

The well-ordered set $(\omega^{\not\downarrow\leq k}, \prec)$, $k \geq 2$.

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$()$ is the $\prec$-minimum element.

For $(j_0, \dots, j_{p-1})$ and $(l_0, \dots, l_{q-1})$ in $\omega^{\not\downarrow\le k}$ with $p, q \ge 1$, define $(j_0, \dots, j_{p-1}) \prec (l_0, \dots, l_{q-1})$ if and only if either

1. $j_{p-1} < l_{q-1}$, or
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Let $\vec{j}_m$ denote the $\prec -m$-th member of $\omega^{\not\downarrow \leq k}$.

For $\vec{l} \in \omega^{\not\downarrow \leq k}$, we let $m_{\vec{l}} \in \omega$ denote *the* m such that $\vec{l} = \vec{j}_m$.

The spaces $\mathcal{E}_k$, $k \geq 2$

$\widehat{\mathbb{W}}_k$ is the image of the function $\vec{l} \mapsto \{m : \vec{j}_m \sqsubseteq \vec{l}\}$, $\vec{l} \in \omega^{\not\downarrow \leq k}$.

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We say that $\widehat{X}$ is an $\mathcal{E}_k$*-tree* if $\widehat{X}$ is a function from $\omega^{\not{\downarrow} \leq k}$ into $\widehat{\mathbb{W}}_k$ such that

(i) For each $m < \omega$, $\widehat{X}(\vec{j}_m) \in [\omega]^{|\vec{j}_m|} \cap \widehat{\mathbb{W}}_k$;

(ii) For all $1 \leq m < \omega$, $\max(\widehat{X}(\vec{j}_m)) < \max(\widehat{X}(\vec{j}_{m+1}))$;

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For each $n < \omega$, the n-th finite aproximation $r_n(X)$ is $X \cap (\{\vec{i}_p : p < n\} \times \mathbb{W}_k)$, where $(\vec{i}_p : p < \omega)$ is the $\prec$-wellordering on $\omega^{\not\leq k}$.

The $\mathcal{E}_k$ are high dimensional Ellentuck spaces

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2. Moreover, each projection of $\mathcal{E}_k$ to levels 1 through j produces a copy of $\mathcal{E}_j$.
3. The trick was finding the right thinning and finite approximation scheme to make Axiom **A.3 (2)** hold. (The Pigeonhole Principle **A.4** was no problem.)

Initial Tukey and Rudin-Keisler structures below $\mathcal{G}_k$, $k \geq 2$

Thm. [D] Let $\mathcal{G}_k$ denote the generic ultrafilter forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes^k}$.

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Remark. The fact that $\mathcal{E}_k$ is dense below any member of $(\mathrm{Fin}^{\otimes k})^+$ provides a simple way of reading off the partition relations for the generic ultrafilter.

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6. Apply the Ramsey-classification theorem for equivalence relations on fronts and analyze $f(\langle \mathcal{G}_k | \mathcal{F} \rangle)$.

Def. A family of finite approximations $\mathcal{F}$ is a *front* on $\mathcal{E}_k$ iff

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Def. A map φ on a front $\mathcal{F} \subseteq \mathcal{AR}$ is called

1. *inner* if for each $a \in \mathcal{F}$, $\varphi(a)$ is a subtree of $\widehat{a}$.
2. *Nash-Williams* if for all pairs $a, b \in \mathcal{F}$, $\varphi(a) \neq \varphi(b)$ implies $\varphi(a) \not\sqsubset \varphi(b)$ (in terms of r).
3. *irreducible* if it is inner and Nash-Williams.

Ramsey-classification Theorem for equivalence relations on fronts

Thm. [D] Let $\mathcal{F}$ be a front on $\mathcal{E}_k$ and $f : \mathcal{F} \to \omega$. Then there exists an $X \in \mathcal{E}_k$ and an irreducible map φ on $\mathcal{F}|X$ such that

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Rem. This is the analogue (extension) of the Pudlák-Rödl Theorem for this space. Further, the canonization maps have the form that $\varphi(a)$ is a projection to some initial segements of the nodes in a.

Thm. [D] Let R be an equivalence relation on some front $\mathcal{F}$ on $\mathcal{E}_k$. Suppose φ and φ' are irreducible maps canonizing R. Then there is an $A \in \mathcal{E}_k$ such that for each $a \in \mathcal{F}|A$, $\varphi(a) = \varphi'(a)$.

For a front $\mathcal{F}$ consisting of the n-th finite approximations $\mathcal{AR}_n$, the canonical equivalence relations are given by projection maps of the form

$$\varphi(a(0), \ldots, a(n-1)) = (\pi_{j_0}(a(0)), \ldots, \pi_{j_{n-1}}(a(n-1))),$$

where $\pi_j(a(i))$ is the projection of $a(i)$ to its first j levels (in the tree $\mathbb{W}_k$).

Basic Cofinal Maps from $\mathcal{G}_k$

Def. Given $Y \in \mathcal{B}_k := \mathcal{G}_k \cap \mathcal{E}_k$, a monotone map $g : \mathcal{B}_k|Y \to \mathcal{P}(\omega)$ is *basic* if there is a map $\hat{g} : \mathcal{AR}|Y \to [\omega]^{<\omega}$ such that

1. (monotonicity) For all $s, t \in \mathcal{AR}|Y$, $s \subseteq t \to \hat{g}(s) \subseteq \hat{g}(t)$;
2. (initial segment preserving) For $s \sqsubset t$ in $\mathcal{AR}|Y$, $\hat{g}(s) \sqsubseteq \hat{g}(t)$;
3. ($\hat{g}$ represents g) For each $V \in \mathcal{B}_k|Y$, $g(V) = \bigcup_{n<\omega} \hat{g}(r_n(V))$.

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Thm. (Basic monotone maps on $\mathcal{G}_k$) [D]
Let $\mathcal{G}_k$ generic for $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$. In $V[\mathcal{G}_k]$, for each monotone function $g : \mathcal{G}_k \to \mathcal{P}(\omega)$, there is a $Y \in \mathcal{B}_k$ such that $g \restriction (\mathcal{B}_k|Y)$ is basic.

Remark. The proofs of the Ramsey-classification Theorem for equivalence relations on fronts and the Basic Cofinal Maps Theorem could be proved using only the Abstract Nash-Williams Theorem, which we originally proved without using **A.3 (2)**.

Infinite dimensional Ellentuck spaces

The sets $[\omega]^k$ are actually uniform barriers (on ω) of finite rank.

Uniform barriers $\mathcal{B}$ (on ω) of any countably infinite rank provide the template for building higher order Ellentuck spaces $\mathcal{E}_{\mathcal{B}}$.

Such spaces $\mathcal{E}_{\mathcal{B}}$ are forcing equivalent to forcings constructed by continuing the process of iteratively constructing ideals built from the ideals $\mathrm{Fin}^{\otimes k}$.

Rather than give all the definitions, we shall now provide an example giving the flavor of these spaces.

The first infinite dimensional Ellentuck space

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$\mathcal{P}(\mathcal{S})/\mathrm{Fin}^{\mathcal{S}}$ is forcing equivalent to $((\mathrm{Fin}^{\mathcal{S}})^+, \subseteq^{\mathrm{Fin}^{\mathcal{S}}})$.

We use the form of $\mathcal{S}$ to make our template structure of finite non-decreasing sequences of natural numbers.

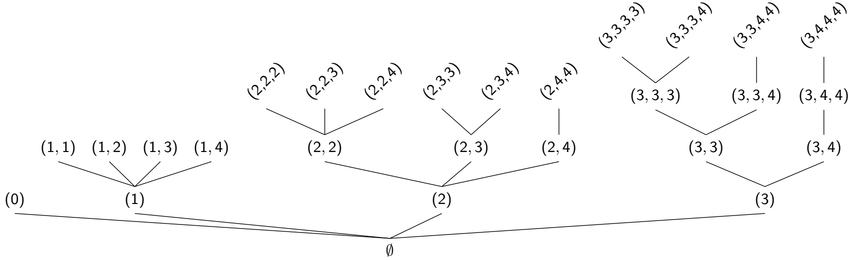


Figure: $\omega^{\not\downarrow \mathcal{S}}$

$() \prec (0) \prec (1) \prec (1,1) \prec (1,2) \prec (2) \prec (2,2) \prec (2,2,2) \prec (1,3) \prec (2,2,3) \prec (2,3) \prec (2,3,3) \prec (3) \prec (3,3) \prec (3,3,3) \prec (3,3,3,3) \prec (1,4) \prec \dots$

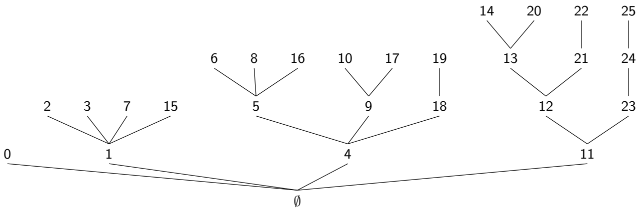
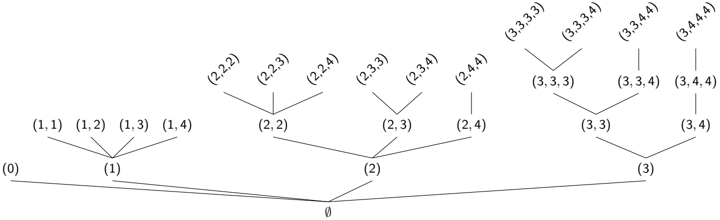


Figure: $\mathbb{W}_{\mathcal{S}}$



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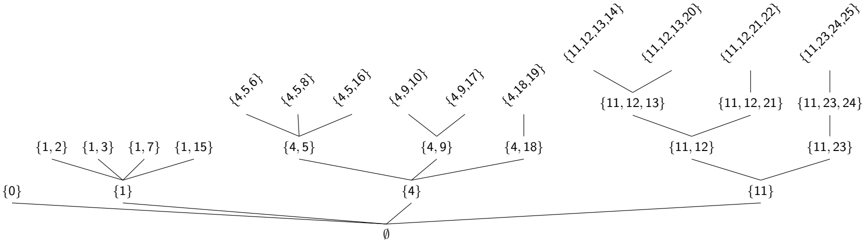

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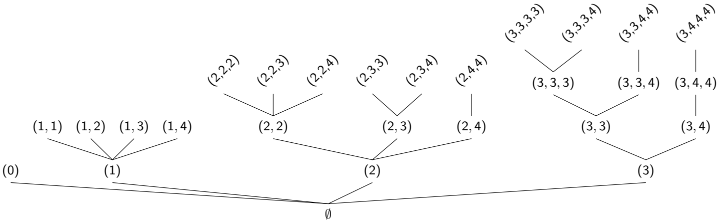
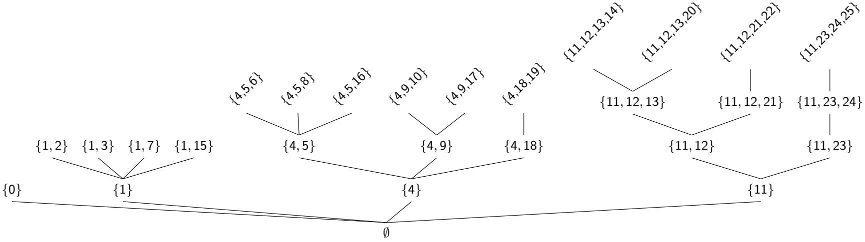


Figure: $\omega^{\not\downarrow \mathcal{S}}$

The space $\mathcal{E}_\mathcal{S}$, for $\mathcal{S}$ the Schreier barrier

$\mathcal{E}_\mathcal{S}$ is the collection of all $X \subseteq \mathbb{W}_\mathcal{S}$ such that

1. for infinitely many k, $\{a \setminus \{k\} : a \in X \text{ and } \min a = k\} \in \mathcal{E}_k$,
2. if $\{a \in X : \min a = k\} \notin \mathcal{E}_k$, then it is empty,
3. The values of the nodes in $\hat{X}$ follow the $\prec$ order.
4. Finitization is recursively induced by the finitizations on the $\mathcal{E}_k$.



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Current work

Let $\mathcal{B}$ be any uniform barrier on ω.

Thm. [D] The space $\mathcal{E}_{\mathcal{B}}$ is a topological Ramsey space.
Forcing with $(\mathcal{E}_{\mathcal{B}}, \subseteq^{\mathrm{Fin}^{\mathcal{B}}})$ is equivalent to forcing with $\mathcal{P}(\mathcal{B})/\mathrm{Fin}^{\mathcal{B}}$.

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3. Special Case: For the Schreier barrier $\mathcal{S}$, the initial Rudin-Keisler structure below the generic ultrafilter $\mathcal{G}_{\mathcal{S}}$ is the ultrapower of $\mathbb{N}$ modulo the projected Ramsey ultrafilter $\pi_1(\mathcal{G}_{\mathcal{S}})$.

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Work in progress: Double checking the proofs, finding the exact initial Tukey structures and RK classes within (Is (2) above exact?).