Metric Arens Irregularity

Fields Undergraduate Research Program, 2014

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Outline

Introduction

What is Arens Irregularity?

- A categorization of Banach Algebras (more precisely, their second-dual spaces) according to properties of two operations defined by R. Arens [1].

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Introduction

What is Arens Irregularity?

- A categorization of Banach Algebras (more precisely, their second-dual spaces) according to properties of two operations defined by R. Arens [1].
- Roughly, a measure of when these two products – left and right – "disagree"
- There are several measures of Arens Irregularity
- Our focus in this project is to introduce a new measure and investigate its properties

Metric Arens Irregularity

E. Hu, R. Hernandez, G. Maierhofer, P. Rao

Introduction and Preliminaries

Initial Geometric Invariant
Motivation
Evaluation for Discrete and Countable Groups
Extension to all Discrete Groups

Examples of NOF and OF sets

Tree Complexity

A new Geometric Invariant

Definition

A Banach Space X is a $\mathbb{C}$ vector space with a complete norm $\|\cdot\|$ In other words,

- $\|x\| \geq 0$ for all $x \in X$
- $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$
- $\|\alpha x\| = \|\alpha\|\|x\|$ for all $\alpha \in \mathbb{C}, x \in X$
- $d(x,y) := \|x-y\|(x, y \in X)$ defines complete metric

[4]

Definition

A Banch Algebra $\mathcal{A}$ is a Banach space with an associative product such that $\|x \star y\| \leq \|x\|\|y\|$ for all x and y in $\mathcal{A}$ [4]

Example

$C[0,1]$, the space of all continuous functions with the domain [0,1], with point-wise product of functions, is a Banach Algebra.[4]

Preliminaries
Biduals

Definition

Given a normed space X, we define X^* to be the space of all bounded linear functionals on X with the norm
$\|f\| = \sup_{\|x\|=1} |f(x)|$.
Then X^* is a Banach Space.
If $f \in X^*$ and $x \in X$ then $f(x)$ is denoted as $\langle f, x \rangle$ [4]

- We could keep defining these, calling the second-dual space of X by $X^{**} = (X^*)^*$.
- There is a canonical embedding of X into X^{**}.

Definition

For $m, n \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$, and $a, b \in \mathcal{A}$, we have the following:
(Left)

$$\langle m \,\Box\, n, f \rangle = \langle m, n \,\Box\, f \rangle \,, \tag{1}$$
$$\langle n \,\Box\, f, a \rangle = \langle n, f \,\Box\, a \rangle \,, \tag{2}$$
$$\langle f \,\Box\, a, b \rangle = \langle f, ab \rangle \tag{3}$$

(Right)

$$\langle m \diamond n, f \rangle = \langle n, f \diamond m \rangle \,, \tag{4}$$
$$\langle f \diamond m, a \rangle = \langle m, a \diamond f \rangle \,, \tag{5}$$
$$\langle a \diamond f, b \rangle = \langle f, ba \rangle \tag{6}$$

Preliminaries

$l_1(G)$ and $l_\infty(G)$

Definition

$l_1(G) = \{f : G \to \mathbb{C} | \sum_{g \in G} |f(g)| < \infty\}$. (All but countably many non-zero values.)

This is a Banach Algebra with convolution given by the formula:

$$fh = \sum_{g \in G} \sum_{ts = g} f(t)h(s)\delta_g.$$

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$l_\infty(G) = \{f : G \to \mathbb{C} | \sup_{g \in G} |f(g)| < \infty\}$.

- $l_\infty(G) = l_1(G)^*$

Preliminaries

βG

Definition

βG is the collection of all ultrafilters on G.

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Definition

An ultrafliter on G ,U, is a collection of subsets of G such that

- $G \in U$,
- $A \in U$ and $A \subset B \Rightarrow B \in U$,
- $A, B \in U \Rightarrow A \cap B \in U$, and
- for every $A \subset G$ either $A \in U$ or $G \setminus A \in U$.

Preliminaries

βG

- We can define two different (assosiative) operations on βG and think of this space as a subset of the unit sphere of $l_\infty(G)^*$ with each of the Arens products.

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- We can define two different (assosiative) operations on βG and think of this space as a subset of the unit sphere of $l_\infty(G)^*$ with each of the Arens products.
- This will allow us to perform calculations using ultrafilters to conclude the results of interest.

Preliminaries
Operations on βG

Definition

We define Ultrafilter Addition, from the left $(+_1)$ and from the right $(+_2)$ the following way, for $U, V \in \beta G$:

$$U +_1 V = \{A \subseteq G | \{g \,|\, Ag^{-1} \in U\} \in V\}$$

$$U +_2 V = \{A \subseteq G | \{g \,|\, g^{-1}A \in V\} \in U\}$$

- $Ag^{-1} = \{mg^{-1} | m \in A\}$
- The resulting additions are again ultrafilters (operation is closed)
- This operation is associative

Embedding Properties

We can carry these operations to $l_\infty(G)^*$.
In fact, we have:

- $U +_1 V = U \square V$,
- $U +_2 V = U \diamond V$, restricted to subsets of G.
- $(\beta G, +_1) \to (l_1(G)^{**}, \square)$ is an injective map whose image is contained in the unit sphere.
- The same result holds for $+_2$ and $\diamond$.

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This trick will make it possible to use the nice properties of ultrafilters for evaluating products of functionals in the bidual.

Preliminaries

βG

- In order to see how different the Arens products are, we'll think of ultrafilters as finitely additive probability measures that only attains the values zero and one.
- The objective is to calculate the geometric invariant using this measures.
- Which sets allow us to distinguish the two products?

Definition

$X \subset G$ is *O.F.* iff there are $Y, Y' \subset G$ such that the sets

$$Y' \cap (\cap_{f \in F} Xf^{-1})$$

and

$$Y \cap (\cap_{h \in H} h^{-1} X^c)$$

are both infinite, for every finite sets $F \subset Y, H \subset Y'$. (Observe that, since $X \subset G$, we are considering Xf^{-1} and $h^{-1}X^c$ to be the translate sets of X and X^c in G, respectively.)

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O.F. Sets

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Theorem

A set $X \subset G$ is O.F. iff there exist $U, V \in \beta G$ such that

$$U \square V(X) - U \diamond V(X) \neq 0.$$

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To start speaking the language of Arens Irregularity, we introduce the following notions –

Definition

Z_l and Z_r are the left and right topological centers, respectively, described by:

$$Z_l\,(\mathcal{A}^{**}) := \{X \in \mathcal{A}^{**} \,|\, X \,\square\, Y = X \diamond Y \;\forall\, Y \in \mathcal{A}^{**}\}, \tag{7}$$

$$Z_r\,(\mathcal{A}^{**}) := \{X \in \mathcal{A}^{**} \,|\, Y \,\square\, X = Y \diamond X \;\forall\, Y \in \mathcal{A}^{**}\}, \tag{8}$$

Definition (Classifications of Arens Regularity)

A Banach Algebra $\mathcal{A}$ is said to be:

1. **Arens Regular** iff $Z_l(\mathcal{A}^{**}) = Z_r(\mathcal{A}^{**}) = \mathcal{A}^{**}$, or equivalently, iff for all $m, n \in \mathcal{A}^{**}$, $m \Box n = m \diamond n$.
2. **Left Strongly Arens Irregular** iff $Z_l(\mathcal{A}^{**}) = \mathcal{A}$.
3. **Right Strongly Arens Irregular** iff $Z_r(\mathcal{A}^{**}) = \mathcal{A}$.
4. **Strongly Arens Irregular** iff $\mathcal{A}$ is LSAI and RSAI

Outline

Motivation

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- LSAI/RSAI, SAI and AR give us somewhat qualitative labels
- There exist algebras that are neither SAI nor AR
- There are various algebras that have proven difficult to categorize with the traditional definitions
- We'd like to see a number to measure Arens Ir/regularity

Definition

We define the **Initial Geometric Arens Irregularity** measure in the following way:

$$\mathfrak{G}_1(\mathcal{A}) = \sup_{m,n \in B_{\mathcal{A}^{**}}} \|m \,\square\, n - m \diamond n\|$$

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Theorem (Properties of the Geometric Invariant)

We see that:

- $\mathfrak{G}_1(\mathcal{A})$ *lies in the interval* $[0, 2]$
- $\mathfrak{G}_1$ *is an isometric invariant*
- $\mathfrak{G}_1(\mathcal{A}) = 0 \leftrightarrow \mathcal{A}$ *is Arens Regular*
- $\mathfrak{G}_1(\mathcal{A}) \geq \mathfrak{G}_1(\mathcal{A}_0)$ *if* $\mathcal{A}_o \subseteq \mathcal{A}$ *where* $\mathcal{A}_o$ *is an algebra*
- *We can make use of ultrafilters to bound* $\mathfrak{G}_1$

Outline

Evaluation for Disc. and Countable Groups

Lemma

There exists an O.F. set in every discrete group.

Theorem

Let G be a countable and discrete group. Then $\mathfrak{G}_1(l_1(G)) = 2$.

Proof Sketch.

Consider a pair of non-commutative ultrafilters arising from an OF set a la theorem 12, and consider the characteristic function of the OF set X minus that of its complement, which is in the unit ball. Linearity properties ensure a value of 2, which is the max, and the sup is evaluated. □

Outline

Extension to all Discrete Groups

Lemma

Every infinite group has a countable subgroup

Theorem

$\mathfrak{G}_1(l_1(G)) = 2$, *for* G *a discrete and infinite group.*

Proof Sketch.

This follows from lemma 19 (OF $\leftrightarrow$ non-com. UFs) and properties of our invariant, theorem 16. □

The Tarski Monster Group

Looking at the geometric invariant

Definition

The *Tarski monster group* is an infinite group G, in which every subgroup $H \leq G$ is finite, in fact, a cyclic subgroup of prime order p.

Definition

An *amenable* group is a locally compact topological group with a mean function, which is invariant under translation.

Theorem (Previously known)

(Forrest) If G contains an infinite amenable subgroup $\Rightarrow$ $\mathfrak{G}_1(l_1(G)) = 2$.

The Tarski Monster Group

Looking at the geometric invariant

- The Tarski monster group does not contain any infinite subgroups, hence cannot conclude $\mathfrak{G}_1(l_1(G)) = 2$ from previous results.
- However, our result pushes $\mathfrak{G}_1(l_1(G)) = 2$ beause the group is discrete and countable.

Even/Odd Cardinality Sets in the Boolean Group

NOF set

Definition

The *Boolean group* $(\mathbb{B}, \triangle)$, is the group consisting of finite sequences of the natural numbers $[\mathbb{N}]^{<\aleph_0}$, together with the binary operation of the symmetric difference $\triangle$.

Illustration.

Let S consist of the set of all subsets of $\mathbb{B}$ with even cardinality. As before, we look at the ultrafilter products $U \square V$, $U \diamond V$:

$$U \square V(S) = \{n | S \triangle n \in U\} \in V$$

$$U \diamond V(S) = \{n | S \triangle n \in V\} \in U$$

□

Even/Odd Cardinality Sets in the Boolean Group (continued)

NOF set

Illustration(continued).

$$U\Box V(S) = \{n|S\triangle n \in U\} \in V$$

$$U \diamond V(S) = \{n|S\triangle n \in V\} \in U$$

We make the general observation that parity within the Boolean group is preserved:
$\text{card}(a\triangle b) = \text{card}(a) + \text{card}(b) - 2 \cdot \text{card}(a \cap b), \forall a, b \in \mathbb{B}$.
Thus, in a similar manner to the evens/odds preserving parity, we have $U\Box V(S) = U \diamond V(S)$, and same for S'. □

Pattern Matching

OF set

Definition

We say that a set X is a *pattern* with respect to infinite sequences Z, Z', if $\exists x \in X, x' \in X^c$ such that $x \in \cap_{i=1}^{n} X z_i^{-1}, x' \in \cap_{i=1}^{n} {z'}_i^{-1} X^c$, where $z_i \in Z, z_i' \in Z'$, $\exists$ infinite $n \in \mathbb{N}$.

Illustration.

Equivalent to definition of an OF set, but this provides a nice visual illustration of an OF set.

$X = \{$ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18... $\}$

where $Z = \{1, 2, 4, 6, 9, ...\}$, blue corresponds to the intersection, red corresponds to the translates. □

Introduction to Tree Approach

- Would like to work through a descriptive set theoretic approach to understanding the complexity of OF sets
- Uses tools such as trees associated to sets, rank of trees associated to sets, Borel sets, etc.
- Helpful tool to provide a visual representation of such sets

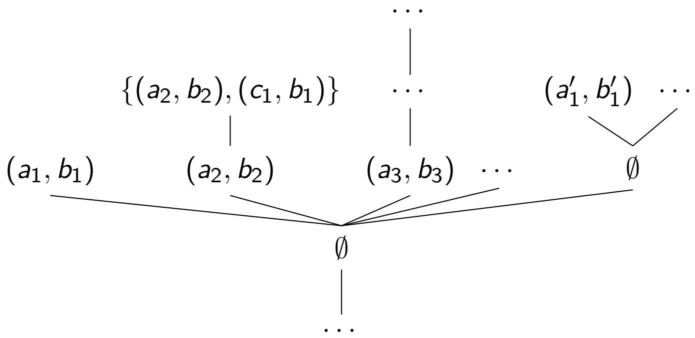
Background in Descriptive Set Theory

Definition

- A *tree* on a set X is a subset $T \subseteq X^{<\mathbb{N}}$ closed under initial segments, i.e., if $t \in T$ and $s \subseteq t$, then $s \in T$.
- The *rank* of a tree is defined as $\rho(T) = \limsup\{\rho(\bar{X}) + 1 | \bar{x} > x\}$, where x is the minimal node, rank of the node is defined similarly..
- A *well-founded* tree is one with no infinite branches. Similarly, an *ill-founded* tree is one with at least one infinite branch.
- We denoted the *associated tree* to a set X by T_X.

Linkage between Trees and Sets

- Use tree building algorithm to build tree.
- Illustration of a particular tree:



Tree Algorithm

Lemma

For any set X, we can construct a tree associated to set X through a tree building algorithm.

Proof.

We introduce a new algorithm to construct a tree in the following manner:

1. Begin with the empty set, $\emptyset$.
2. For each of the following levels $n \in \mathbb{N}$, add a node (f_n, h_n) to each current sequence $\{(f_l, h_l)\}_{l=1}^{n-1}$ in the tree, where $h_n \in \bigcap_{f_i \in F \subset Y} Xf_i^{-1}$ and $f_n \in \bigcap_{h_i \in H \subset Y'} h_i^{-1} X^c$. We stop building the sequence at level n if $\bigcap_{f_i \in F \subset Y} Xf_i^{-1} = \emptyset$ or $\bigcap_{h_i \in H \subset Y'} h_i^{-1} X^c = \emptyset$.

□

Preliminary Results

Lemma

If a tree T_X has an infinite branch, then the set X corresponding to the tree is an OF set.

Sketch of Proof.

Through the algorithm we can see that if we have an infinite branch, we have an infinite Y, Y' under which the intersections are infinite, which means the set is OF. □

Complexity Analysis

Definition

We define $\Sigma_1^1 = \{A \subseteq \mathbb{B} |$ there is a Borel set $B \subseteq \mathbb{B} \times \mathbb{B}$ s.t. $A = \text{proj}_X(B)\}$.

These are the set of analytic sets, which are the continuous image of a Polish space. We would like to show that the set of trees associated to NOF sets is Π_1^1-complete, so that the set of OF sets is Σ_1^1-complete, which gives us an understanding of the complexity of the OF sets.

Successor Ordinal

For simplicity, we first examine the Boolean group $(\mathbb{B}, \triangle)$.

Lemma

For any associated tree T_X associated to a set $X \in \mathbb{B}$, we can find an associated tree $T_{\widetilde{X}}$ associated to a set $\widetilde{X}$ such that if rank$(T_X) = \alpha$, then rank$(T_{\widetilde{X}}) = \alpha + 1$. (Successor Ordinal)

Sketch of proof.

The general idea, is we decompose the Boolean group into two distinct infinite subgroups: $\mathbb{B} = G_0 \oplus G_1$, where $G_0, G_1 \cong \mathbb{B}$. We then look at the image of the set X under the isomorphism $\psi : [\mathbb{N}]^{<\aleph_0} \to [A_0]^{<\aleph_0}$, which is $\psi[X]$. We then form the set $\widetilde{X} = \bigcup_{z_i \in Z}(\psi[X] \triangle z_i)$, and we show that $T_{\widetilde{X}} \supseteq \widetilde{T_X}$, where $\widetilde{T_X}$ are infinitely many copies of $T_{\psi[X]}$ glued with $\emptyset$ replaced with $(z_i, z_i'), z_i \in Z_0 \subset G_1, z_i' \in Z_1 \subset G_0$. Since $\rho(\widetilde{T_X}) = \alpha + 1, \rho(T_{\widetilde{X}}) \geq \alpha + 1$. □

Limit Ordinal

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Lemma

For any limit ordinal of rank α, we can find sets X_i such that $rank(T_{X_i}) = \alpha_i$ such that if we look at $X' = \cup X_i$, $rank(T_{X'}) = \alpha$. (Limit Ordinal)

Sketch of proof.

Similar idea to successor ordinal, except now we decompose $\mathbb{B} = \bigoplus_{i=1}^{\infty} G_i$, $G_i \cong \mathbb{B}$. Take again the isomorphism $\psi_i : [\mathbb{N}]^{<\aleph_0} \to [A_i]^{<\aleph_0}$. Looking at $\widetilde{X} = \bigcup_i^{\infty} \psi_i[X_i]$, we show that $T_{\widetilde{X}} \supseteq \widetilde{T_X}$, where $\widetilde{T_X}$ consists of all the $T_{\psi_i[X_i]}$ glued together. Then, $\rho(T_{\widetilde{X}}) \geq \rho(\widetilde{T_X}) = \alpha$. $\square$

Trees of Arbitrary Rank

Theorem

For any $\alpha < \omega_1$, there exists trees T_X, associated to the set X such that rank$(T_X) = \alpha$.

Sketch of proof.

Proof by contradiction. Suppose $\exists \alpha$, successor ordinal such that there does not exist a NOF set with tree with rank α, and suppose the largest is $\beta = \alpha - 1$. Then by the successor ordinal, we produce a NOF set with tree with rank $\beta + 1 = \alpha$. Contradiction, similar case with limit ordinal. □

Significance of Result

- In particular, this shows we can find trees associated to sets of any rank $\alpha \leq \omega_1$.
- This shows that the not OF sets are Π_1^1-complete.
- Thus, we have that the sets which are on fire are Σ_1^1-complete.

Fr(G), relationship to WAP(G), tree construction

Definition

We define the *freezing algebra* $\mathrm{Fr}(G)$ to be the following:
$\mathrm{Fr}(G) = \{f \in PG : \forall U, V \in \beta G, U \Box V(f) = U \diamond V(f)\}$

Definition

We also have the set of weakly aperiodic functions:
$\mathrm{WAP}(G) = \{f \in \ell_1^* : \forall U, V \in \ell_1^{**}, U \Box V(f) = U \diamond V(f)\}$

We have that $\mathrm{WAP}(G) \cap PG \subseteq \mathrm{Fr}(G)$, since $\beta G \subseteq \ell_1^{**}$. We can use the complexity of the trees to study more properties of $\mathrm{WAP}(G)$ through rank; for example, by asking at which rank do the measures fail to commute in $\mathrm{WAP}(G)$.

Weakness of $\mathfrak{G}_1$

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Previously

There exist banach algebras $\mathcal{A}, \mathcal{B}$ such that

$$\mathfrak{G}_1(\mathcal{A}) = 0, \mathfrak{G}_1(\mathcal{B}) = 2.$$

Weakness of $\mathfrak{G}_1$

Previously

There exist banach algebras $\mathcal{A}, \mathcal{B}$ such that

$$\mathfrak{G}_1(\mathcal{A}) = 0, \mathfrak{G}_1(\mathcal{B}) = 2.$$

Question

Does there exist a banach algebra $\mathcal{C}$ such that

$$\mathfrak{G}_1(\mathcal{C}) \in (0, 2)?$$

Weakness of $\mathfrak{G}_1$

Definition

Let G be a discrete group and $c \in \mathbb{R}, c \geq 1$, then define

$$l_1(G, c) = \{f : G \to \mathbb{C} : \|f\|_c = c \sum_{g \in G} |f(g)| < \infty\}.$$

Weakness of $\mathfrak{G}_1$

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$l_1(G,c) = l_1(G,d)$ as spaces and $l_1(G,c) \cong l_1(G,d)$

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$l_1(G,c) = l_1(G,d)$ as spaces and $l_1(G,c) \cong l_1(G,d)$

$$\|f\|_{c,\star} = \sup_{x \in l_1(G,c), \|x\|_c = 1} |f(x)|$$
$$= \sup_{x \in l_1(G), \|x\| = 1/c} |f(x)|$$
$$\therefore \|f\|_{c,\star} = \frac{1}{c}\|f\|_\star.$$

Similarly $\|m\|_{c,\star\star} = c\|m\|_{\star\star}$.

Weakness of $\mathfrak{G}_1$

$$\mathfrak{G}_1(l_1(G,c)) = \sup_{\|m\|_{c,\star\star}=\|n\|_{c,\star\star}=1} \|m \,\square\, n - m \,\diamond\, n\|_{c,\star\star} =$$

$$= \sup_{\|m\|_{\star\star}=\|n\|_{\star\star}=1/c} c\|m \,\square\, n - m \,\diamond\, n\|_{\star\star} =$$

$$= \frac{1}{c^2} \sup_{\|m\|_{\star\star}=\|n\|_{\star\star}=1} c\|m \,\square\, n - m \,\diamond\, n\|_{\star\star}$$

$$\mathfrak{G}_1(l_1(G,c)) = \frac{1}{c}\mathfrak{G}_1(l_1(G))$$

$$= \frac{2}{c}$$

A new Geometric Invariant

Definition

Let $\mathcal{A}$ be a Banach algebra, then define

$$\mathfrak{G}_2(\mathcal{A}) = \sup_{\mathcal{B} \cong \mathcal{A}} \mathfrak{G}_1(\mathcal{B})$$

$$:= \sup\{x \in [0,2] : \exists \mathcal{B} \cong \mathcal{A} \text{ s.t. } \mathfrak{G}_1(\mathcal{B}) = x\}$$

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- $\mathfrak{G}_2(\mathcal{A}) \in [0,2]$

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- $\mathfrak{G}_2(\mathcal{A}) \in [0,2]$
- $\mathfrak{G}_2(l_1(G,c)) = 2$

$$l_1(G,c) \cong l_1(G,1) = l_1(G)$$

$$\mathfrak{G}_1(l_1(G)) = 2$$

A new Geometric Invariant

A new Geometric Invariant

Proposition

Let $\mathcal{A}$ be a Banach algebra, then

$$\mathfrak{G}_2(\mathcal{A}) = 0 \iff \mathcal{A} \text{ is Arens Regular.}$$

A new Geometric Invariant

Proposition

Let $\mathcal{A}$ be a Banach algebra, then

$$\mathfrak{G}_2(\mathcal{A}) = 0 \iff \mathcal{A} \text{ is Arens Regular.}$$

Lemma

Let $\mathcal{A}, \mathcal{B}$ be two Banach algebras then

$$\mathcal{A} \cong \mathcal{B} \implies \mathcal{A}^{\star\star} \cong \mathcal{B}^{\star\star}$$

A new Geometric Invariant

Proposition

Let $\mathcal{A}$ be a Banach algebra, then

$$\mathfrak{G}_2(\mathcal{A}) = 0 \iff \mathcal{A} \text{ is Arens Regular.}$$

Lemma

Let $\mathcal{A}, \mathcal{B}$ be two Banach algebras then

$$\mathcal{A} \cong \mathcal{B} \implies \mathcal{A}^{\star\star} \cong \mathcal{B}^{\star\star}$$

Lemma

Let $\mathcal{A}$ be a Banach algebra, then

$$\mathfrak{G}_1(\mathcal{A}) = 0 \iff \mathcal{A} \text{ is Arens Regular}$$

A new Geometric Invariant

Sufficient to show if $\mathcal{A}, \mathcal{B}$ are Banach algebras and $\phi : \mathcal{A}^{\star\star} \to \mathcal{B}^{\star\star}$ is an isomorphism, then

$$\mathcal{A} \text{ Arens Regular } \overset{!}{\Longrightarrow} \mathcal{B} \text{ Arens Regular}$$

A new Geometric Invariant

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If $\mathcal{A}$ is Arens Regular, then we have for all $X, Y \in \mathcal{B}^{\star\star}$

$$\begin{aligned} \phi^{-1}(X) \,\Box_{\mathcal{A}} \phi^{-1}(Y) &= \phi^{-1}(X) \diamond_{\mathcal{A}} \phi^{-1}(Y) \\ \Longrightarrow \phi^{-1}(X \,\Box_{\mathcal{B}} Y) &= \phi^{-1}(X \diamond_{\mathcal{B}} Y) \\ \Longrightarrow X \,\Box_{\mathcal{B}} Y &= X \diamond_{\mathcal{B}} Y \end{aligned}$$

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If $\mathcal{A}$ is Arens Regular, then we have for all $X, Y \in \mathcal{B}^{\star\star}$

$$\phi^{-1}(X) \,\Box_{\mathcal{A}} \phi^{-1}(Y) = \phi^{-1}(X) \diamond_{\mathcal{A}} \phi^{-1}(Y)$$
$$\Longrightarrow \phi^{-1}(X \,\Box_{\mathcal{B}} Y) = \phi^{-1}(X \diamond_{\mathcal{B}} Y)$$
$$\Longrightarrow X \,\Box_{\mathcal{B}} Y = X \diamond_{\mathcal{B}} Y$$

$$\Longrightarrow \mathcal{B} \text{ is Arens Regular}$$

Summary

- There is a distinct categorization for getting non-commutative ultrafilters in βG.
- This leads to us being able to evaluate $\mathfrak{G}_{1/2}(l_1(G)) = 2$ for all discrete groups, including the Tarski group.
- Our initial invariant was vulnerable to changes through isomorphisms, so we develop a new, stronger invariant.

Open Questions

- Let $\mathcal{A}$ be a Banach Algebra, and $\mathcal{A}_0$ a subalgebra of $\mathcal{A}$, is it true that
$$\mathfrak{G}_2(\mathcal{A}_0) \leq \mathfrak{G}_2(\mathcal{A})?$$
- Does there exist a Banach Algebra $\mathcal{A}$ such that
$$\mathfrak{G}_2(\mathcal{A}) \in (0,2)?$$
- Are there O.F. sets with algebraic structure, such as subgroups?
- Let G be a locally compact group, is it true that
$$\mathfrak{G}_2(L_1(G)) = 2?$$

For Further Reading I

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Thank you!

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