The Model Theory of C^*-algebras

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August 29, 2014

Abstract

This project report presents conditions on a compact Hausdorff space X for proving $Th(C(X))$ does not have quantifier elimination in the language of metric structures for C^*-algebras. We show that this condition holds in a large class of spaces. In an independent result, it is shown that the class of Hilbert bimodules with a fixed left action are axiomatizable when the underlying vector space and right algebra are considered as sorts.

Contents

1 Introduction

In this project report, we outline two applications of model theory to the study of C^*-algebras. Model theory is a branch of mathematical logic which allows us to study structures in the setting of first-order logic; in this report we focus a framework for *continuous* first-order logic, known as the *model theory for metric structures.* By applying methods and concepts from continuous model theory, we can solve problems in the theory of operator algebras.

In the first section of this report we outline basic definitions and ideas from continuous model theory, following the treatment given in [BY]. We then introduce the first problem considered during the summer project, which involves proving that the class of Hilbert bimodules with a fixed left action is an axiomatizable class in the sense of model theory. We then describe the second problem, in which we determine a partial list of Abelian C^*-algebras whose theories do not admit elimination of quantifiers. We also show in Corollary 4.5 that there is essentially only one Abelian C^*-algebra of real rank zero whose theory admits quantifier elimination; this follows from a result of Eagle and Vignati (see [EagVig]) which says that if two C^*-algebras have real rank zero and both have no isolated points, then their complete theories are equal. We conclude with a short list of open problems.

2 Continuous logic

2.1 Metric structures and languages

Let (M, d) be a complete, bounded metric space.

Definition 2.1. A *predicate* on M is a uniformly continuous function from M^n into a closed, bounded interval in $\mathbb{R}$, where n is a positive integer. A *function* on M is a uniformly continuous function from M^n into M, where n is a positive integer. The positive integer n is called the *arity* of a predicate or function.

Definition 2.2. A *metric structure* $\mathcal{M}$ based on (M, d) is a family of predicates $(R_i : i \in I)$ of predicates on M, a family of functions $(F_j : j \in J)$ on M, and a family $(a_k : k \in K)$ of distinguished elements of M. We denote a metric structure as

$$\mathcal{M} = (M, R_i, F_j, a_k) : i \in I, j \in J, k \in K).$$

Definition 2.3. A *signature* or *language* $\mathcal{L}$ consists of:

- a collection of *predicate symbols* $(P_i : i \in I)$, together with an integer $a(P_i)$ for each $i \in I$ to be interpreted as the arity of a predicate. Furthermore, for each predicate symbol P_i, $\mathcal{L}$ specifies a closed, bounded interval I_{P_i} in $\mathbb{R}$ (to be interpreted as the range of the predicate) and a modulus of uniform continuity Δ_{P_i} (to be interpreted as the modulus of uniform continuity of a predicate);
- a collection of *function symbols* $(f_j : j \in J)$, together with an integer $a(f_j)$ for each $j \in J$ to be interpreted as the arity of a function. Furthermore, for each function symbol f_j, $\mathcal{L}$ specifies a modulus of uniform continuity Δ_{f_j} (to be interpreted as the modulus of uniform continuity of a function);
- a collection of *constant symbols* $(c_k : k \in K)$; and
- a non-negative real number $D_{\mathcal{L}}$, to be interpreted as a bound on the diameter of the underlying metric space (M, d) of a metric structure $\mathcal{M}$.

Given a language $\mathcal{L}$ and a metric structure $\mathcal{M}$ such that the predicate symbols, function symbols and constant symbols of $\mathcal{L}$ correspond exactly to the predicates, functions and distinguished elements which make up $\mathcal{M}$, we say that $\mathcal{M}$ is an *$\mathcal{L}$-structure.* In this situation, we say that each function symbol, predicate symbol and constant symbol is *interpreted in* $\mathcal{M}$; we write $P^{\mathcal{M}}$

for the interpretation of the predicate symbol P in $\mathcal{M}$, $f^{\mathcal{M}}$ for the interpretation of the function symbol f in $\mathcal{M}$, and $c^{\mathcal{M}}$ for the interpretation of the constant symbol c in $\mathcal{M}$. We also sometimes denote the metric d associated to $\mathcal{M}$ by $d^{\mathcal{M}}$ in order to distinguish between interpretations of d in different metric structures.

Without loss of generality, we will always assume that a language $\mathcal{L}$ satifies $D_{\mathcal{L}} = 1$ and $I_P = [0,1] \subset \mathbb{R}$ for every predicate symbol $P \in \mathcal{L}$.

Definition 2.4. Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An *embedding* from $\mathcal{M}$ into $\mathcal{N}$ is a metric space isometry

$$\sigma : (M, d^{\mathcal{M}}) \to (N, d^{\mathcal{N}})$$

which satisfies the following:

- Whenever $f \in \mathcal{L}$ is an n-ary function symbol and $a_1, \ldots, a_n \in M$, we have

$$f^{\mathcal{N}}(\sigma(a_1), \ldots, \sigma(a_n)) = \sigma(f^{\mathcal{M}}(a_1, \ldots, a_n)).$$

- Whenever $P \in \mathcal{L}$ is an n-ary predicate symbol and $a_1, \ldots, a_n \in M$, we have

$$P^{\mathcal{N}}(\sigma(a_1), \ldots, \sigma(a_n)) = P^{\mathcal{M}}(a_1, \ldots, a_n).$$

- Whenever $c \in \mathcal{L}$ is a constant symbol, we have

$$c^{\mathcal{N}} = \sigma(c^{\mathcal{M}}).$$

In this case we say that σ *preserves* or *commutes with* the interpretations of the function symbols, predicate symbols and constant symbols of $\mathcal{L}$. An *isomorphism* between $\mathcal{M}$ and $\mathcal{N}$ is a surjective embedding from $\mathcal{M}$ into $\mathcal{N}$. If there exists an isomorphism between $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic* and we write $\mathcal{M} \cong \mathcal{N}$.

An $\mathcal{L}$-structure $\mathcal{M}$ is a *substructure* of another $\mathcal{L}$-structure $\mathcal{N}$ if $M \subseteq N$ and the inclusion map $i : M \hookrightarrow N$ is an embedding of $\mathcal{M}$ into $\mathcal{N}$. In this case we write $\mathcal{M} \subseteq \mathcal{N}$.

2.2 Terms and formulas

In this section we develop the syntax of continuous first-order logic. To this end, fix a language $\mathcal{L}$ for metric structures.

Definition 2.5. The *atomic formulas* of $\mathcal{L}$ are defined inductively as follows:

- The *symbols* of $\mathcal{L}$ include the predicate symbols, function symbols and constant symbols in $\mathcal{L}$; these are referred to as the *non-logical* symbols of $\mathcal{L}$. The remaining symbols are the *logical* symbols of $\mathcal{L}$, which consist of:
  - A symbol d for the underlying metric space of a metric structure $\mathcal{L}$.
  - An infinite set $V_{\mathcal{L}}$ of *variables*.
  - A symbol for each continuous function $u : [0,1]^n \to [0,1]$ of finitely many variables n, where n is a positive integer (these are referred to as the *connectives* of $\mathcal{L}$.
  - The symbols sup and inf, which can be thought of as quantifiers. (In the setting of continous logic, sup acts as a universal quantifier while inf acts (roughly) as an existential quantifier.)

- The *terms* of $\mathcal{L}$ are defined inductively: Each variable and constant symbol is an $\mathcal{L}$-term. Given n $\mathcal{L}$-terms $t_1, \ldots, t_n$ and an n-ary function symbol f in $\mathcal{L}$, $f(t_1, \ldots, t_n)$ is an $\mathcal{L}$-term.
- The *atomic formulas* of $\mathcal{L}$ are all expressions of the form $P(t_1, \ldots, t_n)$ where P is an n-ary predicate symbol in $\mathcal{L}$ and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms. Expressions of the form $d(t_1, t_2)$ for $\mathcal{L}$-terms t_1, t_2 are also atomic formulas. (Note that this is somewhat redundant since we could view the metric symbol d as a binary predicate symbol if we so choose.)

Definition 2.6. The class of *formulas* of $\mathcal{L}$ is the smallest class of expressions in $\mathcal{L}$ satisfying the following:

- All atomic formulas of $\mathcal{L}$ are $\mathcal{L}$-formauls.
- Let $u : [0,1]^n \to [0,1]$ be a continuous function (i.e. u is a connective) and let $\varphi_1, \ldots, \varphi_n$ be $\mathcal{L}$-formulas. Then $u(\varphi_1, \ldots, \varphi_n)$ is an $\mathcal{L}$-formula.
- If φ is an $\mathcal{L}$-formula and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are $\mathcal{L}$-formulas.

Given an $\mathcal{L}$-formula φ, we say that an occurrence of a variable x is *bound* if x lies within a subformula of φ of the form $\sup_x \psi$ or $\inf_x \psi$. If no occurrences of the variable x are bound, we say that x is *free*. (By a *subformula* of φ we mean any $\mathcal{L}$-formula used in the inductive construction of φ; this corresponds exactly to the notion of subformula in the usual first-order setting.) An *$\mathcal{L}$-sentence* is an $\mathcal{L}$-formula which contains no free variables.

Given a term t or a formula φ in $\mathcal{L}$, we write $t(x_1, \ldots, x_n)$ and $\varphi(x_1, \ldots, x_n)$, respectively, to indicate that the free variables occurring in the term or the formula are among the distinct variables $x_1, \ldots, x_n$.

Definition 2.7. An $\mathcal{L}$-formula is *quantifier-free* if it is formed inductively from atomic formulas without using the symbols sup and inf.

2.3 Interpretation of formulas

Now we will develop the semantics of continuous logic. First, let $\mathcal{M}$ be an $\mathcal{L}$-structure with underlying metric space $(M, d^{\mathcal{M}})$, and let $A \subseteq M$. We extend $\mathcal{L}$ to a new language $\mathcal{L}(A)$ by adding a collection of constants $\{c(a) : a \in A\}$ to $\mathcal{L}$. Each new constant symbol $c(a)$ is interpreted in $\mathcal{M}$ as itself (i.e. $c(a)^{\mathcal{M}}) = a$). We often write a instead of $c(a)$ for the constant symbol in $\mathcal{L}(A)$ to be interpreted as $a \in A$.

Now consider the language $\mathcal{L}(M)$ obtained by naming all elements m of M as constant symbols. Given an $\mathcal{L}(M)$-term $t(x_1, \ldots, x_n)$ we can define the interpretation of t in $\mathcal{M}$ as usual (so that t is a function $t^{\mathcal{M}} : M^n \to M$. We can now define the semantics of continuous first-order logic.

Definition 2.8. Let φ be a sentence in the language $\mathcal{L}(M)$. The *value of φ in $\mathcal{M}$* is a real number in the interval $[0,1]$ denoted by $\varphi^{\mathcal{M}}$, defined by induction as follows. (Note that all terms in the following defiinition are terms in which no variables occur.)

- $(d(t_1, t_2))^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$ for any $L(M)$-terms t_1, t_2.
- For any n-ary predicate symbol P of $\mathcal{L}$ and any $\mathcal{L}(M)$-terms $t_1, \ldots, t_n$,

$$(P(t_1, \ldots, t_n))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}).$$

- For any $\mathcal{L}(M)$-sentences $\psi_1, \ldots, \psi_n$ and any continuous function $u : [0,1]^n \to [0,1]$,

$$(u(\psi_1, \ldots, \psi_n))^{\mathcal{M}} = u(\psi_1^{\mathcal{M}}, \ldots, \psi_n^{\mathcal{M}}).$$

- For any $\mathcal{L}(M)$-formula $\psi(x)$,

$$(\sup_x (\psi(x))^{\mathcal{M}} = \sup \{\psi(a)^{\mathcal{M}} : a \in M\}$$

  where the supremum is taken in the interval $[0,1]$.

- For any $\mathcal{L}(M)$-formula $\psi(x)$,

$$(\inf_x (\psi(x))^{\mathcal{M}} = \inf \{\psi(a)^{\mathcal{M}} : a \in M\}$$

  where the infimum is taken in the interval $[0,1]$.

Definition 2.9. Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}(M)$-formula. We let $\varphi^{\mathcal{M}}$ denote the function $M^n \to [0,1]$ defined by

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = (\varphi(a_1, \ldots, a_n))^{\mathcal{M}}$$

where $a_1, \ldots, a_n$ are in M. Two formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ in a language $\mathcal{L}$ are *logically equivalent* if

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \psi^{\mathcal{M}}(a_1, \ldots, a_n)$$

for every $\mathcal{L}$-structure $\mathcal{M}$ and every n-tuple of elements $a_1, \ldots, a_n \in M$.

Definition 2.10. Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula. An $\mathcal{L}$-*condition* E is a formal expression of the form $\varphi(x_1, \ldots, x_n) = 0$. We say that the $\mathcal{L}$-condition E is *closed* if φ is an $\mathcal{L}$-sentence. We write $E(x_1, \ldots, x_n)$ to indicate that E has the form $\varphi(x_1, \ldots, x_n) = 0$.

Suppose E is the $\mathcal{L}(M)$-condition $\varphi(x_1, \ldots, x_n) = 0$ and let $a_1, \ldots, a_n \in M$. We say E is *true of* $a_1, \ldots, a_n$ *in* M, and write $\mathcal{M} \models E[a_1, \ldots, a_n]$, if $\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = 0$. If E_1 and E_2 are the $\mathcal{L}$-conditions $\varphi_1(x_1, \ldots, x_n) = 0$ and $\varphi_2(x_1, \ldots, x_n) = 0$, respectively, we say that E_1 and E_2 are *logically equivalent* if

$$\mathcal{M} \models E_1[a_1, \ldots, a_n] \text{ iff } \mathcal{M} \models E_2[a_1, \ldots, a_n]$$

holds for every $\mathcal{L}$-structure $\mathcal{M}$ and every n-tuple of elements $a_1, \ldots, a_n \in M$.

2.4 Model theory

We now briefly describe some basic model-theoretic notions. Fix a language $\mathcal{L}$ for metric structures.

Definition 2.11. An $\mathcal{L}$-*theory* T is a set of closed $\mathcal{L}$-conditions. If T is an $\mathcal{L}$-theory and $\mathcal{M}$ is an $\mathcal{L}$-structure, we say that $\mathcal{M}$ *is a model of* T, and write $\mathcal{M} \models T$, if $\mathcal{M} \models E$ for every condition E in T. We write $\mathrm{Mod}_{\mathcal{L}}(T)$ for the class of all $\mathcal{L}$-structures that are models of T (or we write $\mathrm{Mod}(T)$ if the language is clear from context).

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The *theory of* $\mathcal{M}$, denoted $\mathrm{Th}(\mathcal{M})$, is the set of all closed $\mathcal{L}$-conditions true in $\mathcal{M}$. If an $\mathcal{L}$-theory T is of the form $\mathrm{Th}(\mathcal{M})$ for some $\mathcal{L}$-structure $\mathcal{M}$, then we say that T is *complete*. Hence we sometimes refer to $\mathrm{Th}(\mathcal{M})$ as the *complete theory* of $\mathcal{M}$.

Let T be an $\mathcal{L}$-theory and let E be a closed $\mathcal{L}$-condition. We say E is a *logical consequence of* T, and write $T \models E$, if $\mathcal{M} \models E$ holds for every model $\mathcal{M} \models T$.

Definition 2.12. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We say that $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, and write $\mathcal{M} \equiv \mathcal{N}$, if $\varphi^{\mathcal{M}} = \varphi^{\mathcal{N}}$ for every $\mathcal{L}$-sentence φ. Note that two $\mathcal{L}$-structures are elementarily equivalent if and only if their complete theories are equal.

If $M \subseteq N$ (where $(M, d^{\mathcal{M}})$ and $(N, d^{\mathcal{N}})$ are the underlying metric spaces of $\mathcal{M}$ and $\mathcal{N}$, respectively), we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, and write $\mathcal{M} \preceq \mathcal{N}$, if whenever $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula and $a_1, \ldots, a_n \in M$, we have

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(a_1, \ldots, a_n).$$

When this holds, we also say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

A function $F : \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* of $\mathcal{M}$ into $\mathcal{N}$ if whenever $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula and $a_1, \ldots, a_n \in M$, we have

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(F(a_1), \ldots, F(a_n)).$$

Note that, when $M \subseteq N$, $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ if and only if the inclusion map $i : M \hookrightarrow N$ is elementary, in the sense described above.

3 Axiomatization of Hilbert bimodules

3.1 Hilbert bimodules

In this section we define Hilbert bimodules, which are C^*-algebraic objects susceptible to attack by model-theoretic techniques. First we give a few preliminary definitions.

Definition 3.1. Let X be a vector space and let $\mathfrak{B}$ be a C^*-algebra. A $\mathfrak{B}$*-valued positive sesquilinear form on* X is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathfrak{B}$ such that $\langle \cdot, \cdot \rangle$ is linear in the second component, conjugate linear in the first component, and $\langle x, x \rangle \geq_{\mathfrak{B}} 0$ for all $x \in X$. If furthermore $x \in X, \langle x, x \rangle$ implies $x = 0$, then we call $\langle \cdot, \cdot \rangle$ a $\mathfrak{B}$*-valued inner product on* X.

Note that if $\langle \cdot, \cdot \rangle$ is a $\mathfrak{B}$-valued positive sesquilinear form on X, then for all $x, y \in X$ we have $\langle x, y \rangle^* = \langle y, x \rangle$. This fact follows from the *polarization identity*, which says

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle x + i^k y, x + i^k y \rangle$$

for all $x, y \in X$.

Proposition 3.1. [PS] *Let X be a vector space, $\mathfrak{B}$ a C^*-algebra, and $\langle \cdot, \cdot \rangle$ a $\mathfrak{B}$-valued inner product on X. Define $||x||_X = \sqrt{||\langle x, x \rangle||_{\mathfrak{B}}}$. Then $|| \cdot ||_X$ is a norm on X.*

Definition 3.2. Let X be a vector space, $\mathfrak{B}$ a C^*-algebra, and $\langle \cdot, \cdot \rangle$ a $\mathfrak{B}$-valued inner product on X. If X is complete with respect to the norm defined in Proposition 2.1, then we say that X is a *Hilbert* $\mathfrak{B}$*-space.*

Example 3.1. Let $X = \mathfrak{B}$ be a C^*-algebra and define a map $\langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ by setting $\langle A, B \rangle = A^* B$ for $A, B \in \mathfrak{B}$. One can then check that $\langle \cdot, \cdot \rangle$ is a $\mathfrak{B}$-valued inner product, and that the $\mathfrak{B}$ is complete with respect to the norm given by $||A||_{\mathfrak{B}} = \sqrt{||\langle A, A \rangle||_{\mathfrak{B}}}$. Thus $\mathfrak{B}$ is a Hilbert $\mathfrak{B}$-space.

Definition 3.3. Let $\mathcal{H}$ be a Hilbert $\mathfrak{B}$-space. If there exists a linear map $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ (where $\mathcal{L}(\mathcal{H})$ denotes the set of linear maps on $\mathcal{H}$) such that ρ is anti-multiplicative (i.e. such that $\rho(ab) = \rho(b)\rho(a)$ for all $a, b \in \mathfrak{B}$) and such that $\langle h, \rho(b)g \rangle = \langle h, g \rangle b$ for all $b \in \mathfrak{B}, g, h \in \mathcal{H}$, then $\mathcal{H}$ is called a *right Hilbert* $\mathfrak{B}$*-module.*

Example 3.2. Consider the Hilbert $\mathfrak{B}$-space X in Example 2.1 and define a map $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ by setting $\rho(B)(A) = AB$ for all $A \in X, B \in \mathfrak{B}$. One can easily check that ρ satifies the requirements in Definition 2.3, so that the collection $(\mathcal{H}, \langle \cdot, \cdot \rangle, \rho)$ forms a right Hilbert $\mathfrak{B}$-module under the action of ρ. (We will abuse notation and simply write $\mathcal{H}$ for the triple $(\mathcal{H}, \langle \cdot, \cdot \rangle, \rho)$ in the case that we have a right Hilbert module.)

The following result gives us a version of the Cauchy-Schwarz inequality for right Hilbert modules.

Proposition 3.2. [PS] *Let $\mathcal{H}$ be a right Hilbert $\mathfrak{B}$-module for some C^*-algebra $\mathfrak{B}$. For all $x, y \in H$ we have $||\langle x, y \rangle||_{\mathfrak{B}} \leq ||x||_{\mathcal{H}} ||y||_{\mathcal{H}}$.*

We would now like to consider linear maps on a Hilbert $\mathfrak{B}$-space $\mathcal{H}$ which act as adjoints in the usual C^*-algebraic sense.

Definition 3.4. Let $\mathcal{H}$ be a Hilbert $\mathfrak{B}$-space. A linear map $T : \mathcal{H} \to \mathcal{H}$ is *adjointable* if there is a linear operator T^* such that $\langle x, Ty \rangle = \langle T^* x, y \rangle$ for all $x, y \in \mathcal{H}$. We denote the set of all adjointable linear maps on $\mathcal{H}$ by $\mathcal{B}_a(\mathcal{H})$.

Note that not all linear maps have an adjoint; indeed, consider the following

Example 3.3. Let $\mathcal{H} = \mathfrak{A} \oplus C([0,1])$, where $\mathfrak{A}$ is the *-subalgebra of $C([0,1])$ which consists of all functions which vanish at 0. Define $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to C([0,1])$ by

$$\langle (f_1, g_1), (f_2, g_2) \rangle = \bar{f}_1 f_2 + \bar{g_1} g_2$$

and define $\rho : C([0,1]) \to \mathcal{L}(\mathcal{H})$ by

$$\rho(h)(f, g) = (fh, gh).$$

One can check that $\mathcal{H}$ is a right Hilbert C([0,1])-module. Now define a linear, continuous map $T : \mathcal{H} \to \mathcal{H}$ by $T(f, g) = (0, f)$ (one can check that this map is indeed linear and continuous).

Claim. *The linear map T is not adjointable.*

Proof. Suppose for a contradiction that T is adjointable. Let $(g_1, g_2) = T^*(0, 1)$, so in particular $g_1 \in \mathfrak{A}$. Then, given any $f \in \mathfrak{A}$, we see that

$$\bar{g_1} f = \langle T^*(0,1), (f, 0) \rangle = \langle (0,1), T(f, 0) \rangle = \langle (0,1), (0, f) \rangle = f.$$

So $\bar{g_1} f = f$ and hence $g_1(x) = 1$ for all $x \; in (0, 1]$. But $g \in \mathfrak{A}$ and so $g(0) = 0$, and so we have a contradiction (since g is continuous). Thus T cannot have an adjoint. □

We are now ready to define the object of interest for this section.

Definition 3.5. Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras and let $\mathcal{H}$ be a right Hilbert $\mathfrak{B}$-module. If there is a *-homomorphism $\lambda : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H})$, then we call $\mathcal{H}$ a *Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule.* (If $\mathfrak{A}$ and $\mathfrak{B}$ are clear from context we will usually refer to $\mathcal{H}$ simply as a *Hilbert bimodule.*)

Notice that a right Hilbert $\mathfrak{B}$-module $\mathcal{H}$ can be viewed as a Hilbert $\mathbb{C}$-$\mathfrak{B}$-bimodule, where the action of $\mathbb{C}$ on $\mathcal{H}$ is simply scalar multiplication by complex numbers.

Example 3.4. Let $\mathcal{H}$ be the right Hilbert $\mathfrak{B}$-module given in Example 2.2. Given $A \in \mathfrak{B}$, define $\lambda(A)B = AB$. Then $\lambda : \mathfrak{B} \to \mathcal{B}_a(\mathcal{H})$ is a unital *-homomorphism, and so $\mathcal{H}$ is a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule.

3.2 Axiomatizability

Definition 3.6. Let $\mathscr{C}$ be a class of $\mathcal{L}$-structures. We say that the class $\mathscr{C}$ is *axiomatizable* if there exists a theory T such that $\mathscr{C} = \mathrm{Mod}_{\mathcal{L}}(T)$. In this case, we say that T is a set of *axioms* for $\mathscr{C}$ in $\mathcal{L}$, or that T is an *axiomatization* of the class $\mathscr{C}$.

We are interested in axiomatizing the class $\mathscr{C}$ of Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodules for a fixed C^*-algebra $\mathfrak{A}$ and a varying C^*-algebra $\mathfrak{B}$. We prove that $\mathscr{C}$ is axiomatizable by constructing an *equivalence of categories* between the classes $\mathscr{C}$ and $\mathrm{Mod}_{\mathcal{L}}(T)$ for some continuous theory T. Explicitly, we want to determine a theory T (in an appropriate language $\mathcal{L}$) such that the following hold:

- For every $A \in \mathscr{C}$ there is a model $\mathcal{M}$ of T determined up to isomorphism.
- For every model $\mathcal{M}$ of T there is some $A \in \mathscr{C}$ such that $\mathcal{M} \cong \mathcal{M}(A)$.
- For any $A, B \in \mathscr{C}$, we can find a bijection between $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\mathcal{M}(A), \mathcal{M}(B))$ (where $\mathrm{Hom}(X, Y)$ denotes the class of morphisms $f : X \to Y$).

The language and axioms presented here will follow the notation given by P. Skoufranis [PS]. We fix a unital C^*-algebra $\mathfrak{A}$ and assume all C^*-algebras $\mathfrak{B}$ will be unital. The language $\mathcal{L}_{\mathfrak{A}}$ for Hilbert $\mathfrak{A}$-$\mathfrak{B}$ modules is the following data:

1. Sorts (H_n, d_n^H) and (B_n, d_n^B) representing the closed balls of radius n for $\mathcal{H}$ and $\mathfrak{B}$ respectively along with injections $\iota_{nm}^H : H_n \to H_m$ and $\iota_{nm}^B : B_n \to B_m$ for each $n < m$[1];
2. the symbols $+_{\mathcal{H}}$, $+_{\mathfrak{B}}, \cdot_{\mathfrak{B}}, {}^*{}_{\mathfrak{B}}$, $-_{\mathcal{H}}$, $-_{\mathfrak{B}}$, $0_{\mathcal{H}}$, $0_{\mathfrak{B}}$, and $1_{\mathfrak{B}}$ for the appropriate sorts;
3. a right action
$$\rho : \mathcal{H} \times \mathfrak{B} \to \mathcal{H}$$
for each appropriate sort;
4. a $\mathfrak{B}$-inner product
$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathfrak{B} \;;$$
5. unary functions representing the action of each element of $\mathfrak{A}$ on $\mathcal{H}$, denoted by $\mu_a : \mathcal{H} \to \mathcal{H}$ for each $a \in \mathfrak{A}$;
6. unary functions representing scalar multiplication of each element of $\mathbb{C}$ on $\mathfrak{B}$, denoted by $\lambda_z : \mathfrak{B} \to \mathfrak{B}$ for each $z \in \mathbb{C}$; and
7. unary functions representing scalar multiplication of each element of $\mathbb{C}$ on $\mathcal{H}$, denoted by $\lambda_z : \mathcal{H} \to \mathcal{H}$ for each $z \in \mathbb{C}$.

The axioms are then as follows. We use the shorthand $a = b$ to mean $d(a, b) = 0$ and suppress '$= 0$' at the end of $\mathcal{L}_{\mathfrak{A}}$-conditions as appropriate.

1. The C^*-algebra axioms for $\mathfrak{B}$ [axioms 1 - 10 of [FHS13]];
2. the axioms of a $\mathbb{C}$-vector space for $\mathcal{H}$ [axiom 1 of [FHS13]];
3. the axioms stating that the injections ι_{nm} behave as embeddings;

[1] We take the shorthand $f : \bar{S} \to S$ to mean that $\mathrm{dom}(f) = \bar{S}$ and $\mathrm{ran}(f) = S$.

4. the axioms setting ρ as the appropriate right ring action on $\mathcal{H}$, *i.e.*,

   i $\sup_{x\in\mathcal{H}}\sup_{a,b\in\mathfrak{B}}\rho(x,a+b)=\rho(x,a)+\rho(x,b)$,

   ii $\sup_{x,y\in\mathcal{H}}\sup_{a\in\mathfrak{B}}\rho(x+y,a)=\rho(x,a)+\rho(y,a)$,

   iii $\sup_{x\in\mathcal{H}}\sup_{a,b\in\mathfrak{B}}\rho(x,ab)=\rho(\rho(x,a),b)$, and

   iv $\rho(x,1)=x$;

5. the axioms defining $\langle\cdot,\cdot\rangle$ as a $\mathfrak{B}$ inner product [2], that is:

   i $\sup_{x,y,z\in\mathcal{H}}\sup_{a\in\mathfrak{B}}\langle x,ya+z\rangle=\langle x,y\rangle a+\langle x,z\rangle$,

   ii $\sup_{x,y\in\mathcal{H}}\langle x,y\rangle=\langle y,x\rangle^*$,

   iii $\sup_{x\in\mathcal{H}}\inf_{a\in\mathfrak{B}}\langle x,x\rangle=y^*y$, and

   iv $\sup_{x,y\in\mathcal{H}}\langle x,\lambda_z y\rangle=z\langle x,y\rangle$ for every $z\in\mathbb{C}$;

6. the axiom making sure the metric d^H is indeed the norm, *i.e.*,

$$\sup_{x\in\mathcal{H}} d^H(x,y)^2=||\langle x-y,x-y\rangle||_{\mathfrak{B}}\ ;$$

   where $||\cdot||_{\mathfrak{B}}:=d^B(\cdot,0)$

7. the axioms for the left action μ_a on $\mathcal{H}$, *i.e.*,

   i $\sup_{x\in\mathcal{H}}\mu_a\mu_b x=\mu_{ab}x$,

   ii $\sup_{x\in\mathcal{H}}\mu_{za}x=\lambda_z\mu_a x$,

   iii $\sup_{x\in\mathcal{H}}\mu_a x+\mu_b x=\mu_{a+b}x$,

   iv $\sup_{x,y\in\mathcal{H}}\mu_a(x+y)=\mu_a x+\mu_a y$,

   v $\sup_{x\in\mathcal{H}}\mu_1 x=x$, and

   vi $\sup_{x,y\in\mathcal{H}}\langle x,\mu_a y\rangle=\langle\mu_{a^*}x,y\rangle$

   for all a and b in $\mathfrak{A}$, and $z\in\mathbb{C}$; and finally the axioms

8. $\sup_{x\in H_1}||x||\dot{-}1$ and

9. $\sup_{x\in H_n}\inf_{y\in H_1}\min(1\dot{-}||x||_{\mathcal{H}},d_n^H(x,\iota_{1n}^H(y)))$,
   where $||\cdot||_{\mathcal{H}}$ means $d^H(\cdot,0)$, to make sure that the injection maps and the sorts behave appropriately.

We set T to be the theory containing the above axioms.

Proposition 3.3. *The class of Hilbert bimodules over unital C^*-algebras with fixed left action and varying right action is axiomatized by the theory T.*

[2]For clarity, we remove ρ from the action.

Proof. Take $\mathscr{C}$ to be the class of all Hilbert $\mathfrak{A}$-$\mathfrak{B}$ bimodules for $\mathfrak{A}$ a fixed unital C^*-algebra and $\mathfrak{B}$ a unital C^*-algebra. The pair $(\mathcal{H}, \mathfrak{B})$ denotes the underlying Hilbert space $\mathcal{H}$ and the right action $\mathfrak{B}$ for an element in $\mathscr{C}$.[3] We then define

$$\begin{aligned} M &: \mathscr{C} \to \mathrm{Mod}(T) \\ &: (\mathcal{H}, \mathfrak{B}) \mapsto \begin{cases} H_n = \{x \in \mathcal{H} : ||x||_{\mathcal{H}} \leq n\}, \\ B_n = \{x \in \mathfrak{B} : ||x||_{\mathfrak{B}} \leq n\}, \text{ and} \\ \text{the symbols interpreted in the given way.} \end{cases} \end{aligned}$$

We wish to show that M is an equivalence of categories. We know that M is a well-defined map by the way we have constructed T. As well, for any $\mathfrak{H}_1 := (\mathcal{H}_1, \mathfrak{B}_1)$ and $\mathfrak{H}_2 := (\mathcal{H}_2, \mathfrak{B}_2)$ in $\mathscr{C}$, if

$$\mathfrak{H}_1 \xrightarrow{\sigma} \mathfrak{H}_2$$

is a homomorphism, then defining

$$\begin{gathered} M(\mathfrak{H}_1) \xrightarrow{M\sigma} M(\mathfrak{H}_2) \text{ by} \\ x \in X_n^{\mathfrak{H}_1} \overset{M\sigma}{\longmapsto} \sigma(x) \in X_n^{\mathfrak{H}_2} \end{gathered}$$

where $X \in \{H, B\}$, we have a bijection

$$\mathrm{Hom}(\mathfrak{H}_1, \mathfrak{H}_2) \xrightarrow{M} \mathrm{Hom}(M(\mathfrak{H}_1), M(\mathfrak{H}_2)) \ .$$

Therefore, to get an equivalence of categories, it suffices to show that, given any $\mathcal{M} \in \mathrm{Mod}(T)$, there is an $\mathfrak{H} \in \mathscr{C}$ such that $M(\mathfrak{H}) \cong \mathcal{M}$.

As the $\iota_{nm}^{\mathcal{M}}$ are injective maps, by taking isomorphisms, we may assume that

$$X_n^{\mathcal{M}} \subseteq X_m^{\mathcal{M}}$$

for all $n \leq m$ and $X \in \{H, B\}$. Let us set

$$\begin{aligned} \mathcal{H} &:= \bigcup_{n<\omega} H_n^{\mathcal{M}} \text{ and} \\ \mathfrak{B} &:= \bigcup_{n<\omega} B_n^{\mathcal{M}} \end{aligned}$$

and we take operations $\mathcal{H}$ and $\mathfrak{B}$ as given by the direct limit. Define $\mathfrak{H} := (\mathcal{H}, \mathfrak{B})$.

Claim. $M(\mathfrak{H}) \cong \mathcal{M}$.

To see the isomorphism holds, we define

$$\begin{aligned} \iota &: \mathcal{M} \to M(\mathfrak{H}) \\ &: x \in H_n^{\mathcal{M}} \mapsto x \in \mathcal{H} \\ &\quad b \in B_n^{\mathcal{M}} \mapsto b \in \mathfrak{B} \end{aligned}$$

[3] If this proof was put into more detail, we should consider the tuple

$$(\mathcal{H}, \mathfrak{B}, \langle \cdot, \cdot \rangle, \mu, \rho)$$

where μ and ρ denote the left and right actions respectively, but this would create clutter for no gain in understanding.

By the construction of the direct limit, ι is an embedding. It remains to show that ι is surjective. By the axiom

$$\sup_{x \in H_n} \inf_{y \in H_1} \min(1 \dot{-} ||x||_{\mathcal{H}}, d_n^H(x, \iota_{1n}^H(y))) ,$$

we have that, for all $n \in \mathbb{N}$,

$$H_n^{\mathcal{M}} = \{x \in \mathcal{H} : ||x||_{\mathfrak{H}} \leq n\}$$

and, by the fact that we have the C^*-algebra axioms, we know

$$B_n^{\mathcal{M}} = \{x \in \mathfrak{B} : ||x||_{\mathfrak{B}} \leq n\} .$$

This tells us that

$$\iota : M(\mathfrak{H}) \xrightarrow{\cong} \mathcal{M} .$$

Therefore, we have that M is an equivalence of categories. □

4 Quantifier elimination for C^*-algebras

4.1 The theory of $C(2^{\mathbb{N}})$

Definition 4.1. Let $\mathcal{L}$ be a language and T be an $\mathcal{L}$-theory. An $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ is *approximable in T by quantifier-free formulas* if for every $\epsilon > 0$ there is a quantifier-free $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ such that

$$|\varphi^{\mathcal{M}}(a_1, \ldots, a_n) - \psi^{\mathcal{M}}(a_1, \ldots, a_n)| \leq \epsilon$$

for every model $\mathcal{M} \models T$ and all n-tuples $a_1, \ldots, a_n \in M$.

An $\mathcal{L}$-theory T *admits quantifier elimination* if every $\mathcal{L}$-formula is approximable in T by quantifier-free formulas.

Recall that a space X is a *Cantor space* if it is homeomorphic to the Cantor set. Note that the Cantor space (which we write as $2^{\mathbb{N}}$) is a zero-dimensional compact Hausdorff space; also note that it has no isolated points.

Proposition 4.1. *The theory of $C(2^{\mathbb{N}})$ admits elimination of quantifiers.*

Sketch of the proof. The set of projections in $C(2^{\mathbb{N}})$ is definable. Moreover, in the case of the Cantor space any $f \in C(2^{\mathbb{N}})$ such that $f = f^*$ (i.e. any self-adjoint element of $C(2^{\mathbb{N}})$) is approximable by functions with finite spectrum.

This tells us $C(2^{\mathbb{N}})$ is separable with a dense set definable in the language of C^*-algebras.

If there is an isomorphism between substructures of $C(2^{\mathbb{N}})$ and any other Abelian C^*-algebra A which satisfies same theory, then a 'big' subset of projections from $C(2^{\mathbb{N}})$ should be mapped in A by the isomorphism, hence we can extend that isomorphism to an embedding.

By Proposition 13.5 in [BY] this implies that $Th(C(2^{\mathbb{N}}))$ admits quantifier elimination. □

Notice that Proposition 4.1 implies that the theory of $C(2^{\mathbb{N}})$ is the only theory of a real rank zero Abelian C^*-algebra which admits quantifier elimination.

4.2 The theory of Peak Spaces

4.2.1 Notation and Definitions

Note: Any time the word function *is used we mean a continuous function.*

Definition 4.2. *We say that a function $f : U \to [0, \infty)$ on a compact Hausdorff space U is a* peak function *provided*

- $\mathrm{sp}(f) = [0, 1]$, *and*
- *the set $\{x \in U : f(x) > 1 - \frac{1}{5}\}$ is connected.*

We say that a space U has the peaking property *if $C(U)$ has a peak function.*

Example 4.1. *Notice the pyramid function*

$$\begin{aligned} p_n &: [-1,1]^n \to [0,1] \\ &: \bar{t} \mapsto \min(p(t_1)\dots, p(t_n)) \end{aligned}$$

has the peaking property for

$$\begin{aligned} p &: [-1,1] \to [0,1] \\ &: t \to \begin{cases} 1+t, & t \leq 0 \\ 1-t, & t \geq 0 \end{cases} . \end{aligned}$$

We set the function

$$\phi(x) := \inf_{f,g} \max(\|x - (f^*f + g^*g)\|, |\|g^*g\| - 1|, |\|f^*f\| - 1|, \|f^*fg^*g\|)$$

to characterize the property that a function x may be split into two orthogonal positive functions of norm one.

4.2.2 Peak Functions Properties in $C(U)$

Suppose U is a compact Hausdorff space and $\alpha : U \to \mathbb{C}$ a peak function.

In this section we will prove $\phi(\alpha)^{C(U)} \neq 0$ (where ϕ is as above).

Roughly speaking what we are trying to say is that α cannot be approximated by the sum of two orthogonal positive functions.

Define the function ψ as follows

$$\psi(f, g, \alpha) = \max\{\|f + g - \alpha\|, |\|f\| - 1|, |\|g\| - 1|, \|fg\|\}$$

Let $f, g : U \to [0, \infty)$ be positive functions of norm 1 and fix

$$A := \{x \in U : \alpha(x) > 1 - \frac{1}{5}\} .$$

We are interested in following result.

Proposition 4.2.

$$\psi(f, g, \alpha) \geq \frac{1}{10}.$$

We will suppose the claim is false and prove a few claims to reach a contradiction. Following claims in this section will suppose $n = 10$.

Claim. *If $x, y \in U$ satisfies $f(x) > 1 - \frac{1}{n}$ and $g(y) > 1 - \frac{1}{n}$ then $x, y \in A$.*

Proof. We will prove it for x and it follows similarly for y.
We know $|f(x) + g(x) - \alpha(x)| < \frac{1}{n}$. Hence $f(x) + g(x) - \frac{1}{n} < \alpha(x)$.
Since $f(x) > 1 - \frac{1}{n}$ and $g(x) \geq 0$, $f(x) + g(x) > 1 - \frac{1}{n}$.
Therefore $1 - 2\frac{1}{n} < \alpha(x)$. By definition, this means $x \in A$. $\square$

Small Claim. *There exists $x, y \in U$ such that $f(x), g(y) > 1 - \frac{1}{n}$.*

Proof. This is true since we are assuming $\psi(f, g, \alpha) < \frac{1}{n}$ and that implies $\|f\| > 1 - \frac{1}{n}$; the same holds for g. $\square$

Proposition 4.3. *If $\psi(f, g, \alpha) < \frac{1}{n}$ then there is a $z \in A$ such that $f(z) = g(z)$.*

Proof. We are still working with $\psi < \frac{1}{n}$. Given this we know $g(x)f(x) < \frac{1}{n}$ for all $x, y \in U$. If $f(x), g(y) > 1 - \frac{1}{n}$ then

$$\begin{aligned} g(x)\left(1 - \frac{1}{n}\right) < g(x)f(x) &< \frac{1}{n} \\ g(x) - \frac{1}{n}g(x) &< \frac{1}{n} \\ g(x) < (1 + g(x))\frac{1}{n} &< 3\frac{1}{n}. \end{aligned}$$

This induces the following inequalities

$$\begin{aligned} g(x) < 3\frac{1}{n} < 1 - \frac{1}{n} < g(y) \\ f(y) < 3\frac{1}{n} < 1 - \frac{1}{n} < f(x). \end{aligned}$$

As we remember, the definition of a peak function implies A is connected, then function $h = f - g$ satisfies $h(x) > 0 > h(y)$. Since $x, y \in A$ and A connected there is a $z \in A$ such that $h(z) = 0$. $\square$

Here is a contradictory result. We are still assuming $\frac{1}{n} = \frac{1}{10}$.

Proposition 4.4. *If $\psi(f, g, \alpha) < \frac{1}{n}$ then there is no $z \in A$ such that $f(z) = g(z)$.*

Proof. Suppose there is one. Then $z \in A$ implies $\alpha(z) > 1 - 2\frac{1}{n}$.
It follows directly that

$$\begin{aligned} 2f(z) &= 2g(z) \\ &= f(z) + g(z) \\ &> \alpha(z) - \frac{1}{n} \\ &> 1 - 2\frac{1}{n} - \frac{1}{n} \\ &= 1 - 3\frac{1}{n} . \end{aligned}$$

Hence

$$f(z)g(z) > \frac{(1-3\frac{1}{n})^2}{4}$$

since

$$\psi(f,g,\alpha) < \frac{1}{n} \text{ implies } f(z)g(z) < \frac{1}{n}$$

and so

$$\frac{1}{n} < \frac{(1-3\frac{1}{n})^2}{4} \ .$$

We have a contradiction and we can conclude z does not exist. □

Now with enough material we come back to prove Proposition 4.2.

Proof of Proposition 4.2 . Suppose $\psi(f,g,\alpha) < \frac{1}{10}$, then Propositions 4.3 and 4.4 hold but they induce a contradiction. □

Corollary 4.1.

$$\phi(\alpha)^{C(U)} \geq \frac{1}{10} > 0.$$

Proof. Directly from Proposition 4.2. □

4.2.3 Volcano Functions $C(U)$

In this section we construct a new function given a peak function in $C(U)$ and make two new ones such that they are orthogonal.

The idea is send the 'peak' of the original function to 0 but preserving the spectrum and use compactness properties of U as well. To illustrate how is this construction think as follows. If we do this to a function in $\mathbb{R}^2 \to \mathbb{R}$ in which its graph is a cone, the new function looks like a volcano.

We should the normalize these functions and make them orthogonal. To do that we take the 'maximum point where graphs intersects' and send it to 0.

Proposition 4.5. *Given $\alpha : U \to \mathbb{C}$, a peak function, we construct a new function $\beta : U \to \mathbb{C}$ which satisfies*

$$\max\{||\beta - (\beta_1 + \beta_2)||, |||\beta_1|| - 1|, |||\beta_2|| - 1|, ||\beta_1\beta_2||\} = 0$$

for positive functions β_1 and β_2 and such that $\mathrm{sp}(\beta) = [0,1]$.

Proof. Fix a point $x_0 \in U$ such that $\alpha(x_0) < 1$. Define

$$\begin{aligned} \upsilon &: U \to \mathbb{C} \\ &: x \mapsto 1 - 2|\alpha(x) - \alpha(x_0)| \ . \end{aligned}$$

Consider the set $\{x \in U : \alpha(x) = \upsilon(x)\}$. This set is finite since $\alpha(x) = \upsilon(x) = t$ when $t = 1 - 2|t - \alpha(x_0)|$. Therefore, it has a maximum θ. The value $\theta < 1$ since the peak p is not in $\{x \in U : \alpha(x) = \upsilon(x)\}$. Consider

$$\beta_1 = \frac{1}{1-\theta}(\alpha \dot{-} \theta) \text{ and}$$
$$\beta_2 = \frac{1}{1-\theta}(\upsilon \dot{-} \theta) ,$$

where $x \dot{-} y := \max\{x - y, 0\}$. We see that $||\beta_1|| = 1 = ||\beta_2||$ since both α and υ are in $[0, 1]$, and attains a maximum at 1. By definition, these functions are positive. Now,

Small Claim. *The product $\beta_1(x)\beta_2(x) = 0$ for all $x \in U$.*

Proof of subclaim. Take any $x \in U$. Assume that $\beta_1(x)\beta_2(x) > 0$ to derive a contradiction. In this case, we must conclude $(\alpha \dot{-} \theta)(x) = \alpha(x) - \theta > 0$ and $(\upsilon \dot{-} \theta)(x) = \upsilon(x) - \theta > 0$. Then

$$0 < (\alpha(x) - \theta)(\upsilon(x) - \theta = \alpha(x)\upsilon(x) - \theta(\alpha(x) + \upsilon(x)) + \theta^2 .$$

Therefore, the discriminant of this quadratic in $\theta(x)$ must be negative. Hence

$$\begin{aligned}(\alpha(x) + \upsilon(x))^2 - 4\alpha(x)\upsilon(x) &< 0\\ \alpha(x)^2 - 2\alpha(x)\upsilon(x) + \upsilon(x)^2 &< 0\\ (\alpha(x) - \upsilon(x))^2 &< 0 .\end{aligned}$$

However, by $\alpha(x) > \theta$, $\alpha - \upsilon \neq 0$. Therefore, we have a contradiction. □

We take $\beta := \beta_1 + \beta_2$ to get the proof of the claim. □

4.2.4 Quantifier Elimination in $Th(C(U))$

Proposition 4.6. *If U is a compact Hausdorff space with a peak function $\alpha : U \to [0, \infty)$ then $C(U)$ does not admit quantifier elimination.*

Proof. By Corollary 4.1, $\phi(\alpha)^{C(U)} \neq 0$. By proposition 4.5, there is a $\beta : U \to [0, \infty)$ with $\mathrm{sp}(\beta) = [0, 1] = \mathrm{sp}(\alpha)$ such that $\phi(\beta)^{C(U)} = 0$. Since $\mathrm{sp}(\alpha) = \mathrm{sp}(\beta)$, if $C(U)$ has quantifier elimination, then by the spectral theorem, we would get $\phi(\alpha)^{C(U)} = \phi(\beta)^{C(U)}$ – a contradiction. □

With this, we have the following result:

Corollary 4.2. *Let X be a compact Hausdorff space with an embedding $\Psi : U \to X$ on a compact U with a peak function $\alpha : U \to \mathbb{C}$ such that $\Psi(U)^\circ \neq \varnothing$ and the set $\Psi(\{x \in U : \alpha(x) > 0\})$ is open. Then $C(X)$ does not admit quantifier elimination.*

Proof. It suffices to show that the function

$$\begin{aligned}\bar{\alpha} &: X \to [0, \infty)\\ &: x \mapsto \begin{cases}\alpha(\Psi^{-1}(x)), & x \in \Psi(U)\\ 0, & x \in X \setminus \Psi(U)^\circ\end{cases}\end{aligned}$$

is a peak function. By the gluing lemma, it suffices to show that, given a point $x \in \Psi(U) \setminus \Psi(U)^\circ$, $\alpha\Psi^{-1}(x) = 0$. Suppose not. Notice first that the set

$$V := \{x \in U : \alpha(x) > 0\}$$

is open. Therefore, $\Psi(V) \subseteq \Psi(U)$ is an open neighbourhood of x. Therefore $x \in \Psi(U)^\circ$ –a contradiction. Furthermore, $[0, 1] = \mathrm{ran}(\alpha) = \mathrm{ran}(\bar{\alpha})$. Finally,

$$\begin{aligned}\{x \in X : \bar{\alpha}(x) > 1 - \frac{1}{5}\} &= \{x \in X : \alpha(\Psi^{-1}(x)) > 1 - \frac{1}{5}\} \\ &= \Psi(\{x \in U : \alpha(x) > 1 - \frac{1}{5}\}) \; .\end{aligned}$$

By the intermediate value theorem this set is connected. Therefore, by proposition 4.6, $\mathrm{Th}(C(X))$ does not admit quantifier elimination. □

4.2.5 Examples for spaces U without quantifier elimination

We see that since n-manifolds may embed an n-cube in the manner proposed in the proposition that

Corollary 4.3. *Given any n-manifold M,* $\mathrm{Th}(C(M))$ *does not have quantifier elimination.*

In fact, proposition 4.6 tells us

Corollary 4.4. *Continuous functions on CW-complexes, simplicial complexes, and the Hawaiian earring does not admit quantifier elimination.*

4.3 Other spaces

Proposition 4.7. *Let X be a compact Hausdorff space with an isolated point x_0. Then* $\mathrm{Th}(C(X))$ *does not eliminate quantifiers.*

Proof. Let $p := \chi_{x_0}$. We have the following facts:

1. $(p, p, 0)$ and $(p, 0, 0)$ are projections with the same spectrum on the space $C(X)^3$.
2. Two projections on $C(X)^3$ can add to $(p, p, 0)$, namely $(p, 0, 0)$ and $(0, p, 0)$.

Therefore we can consider

$$\begin{gathered}\phi(x_1, x_2, x_3) = \inf_{a_1,a_2,a_3,b_1,b_2,b_3} \max_{k=1,2,3} \{||x_k - (a_k^* a_k + b_k^* b_k)||, \\ |\max\{||a_1^* a_1||, ||a_2^* a_2||, ||a_3^* a_3||\} - 1|, |\max\{||b_1^* b_1||, ||b_2^* b_2||, ||b_3^* b_3||\} - 1|\}\end{gathered}$$

If we have quantifier elimination, we expect $\phi(p, 0, 0) = \phi(p, p, 0)$ by the spectral theorem. We see $\phi(p, p, 0) = 0$.

Claim. $\phi(p, 0, 0) \neq 0$.

Proof. Suppose to derive a contradiction that $\phi(p, 0, 0) = 0$. Then, for all $N \in \mathbb{N}$, let a_k^N and b_k^N be positive functions with $a^N = (a_1^N, a_2^N, a_3^N)$ and $b^N = (b_1^N, b_2^N, b_3^N)$ of norms

$$\begin{aligned}|||a^N|| - 1| \le \frac{1}{N} \text{ and} \\ ||b^N|| - 1| \le \frac{1}{N}\end{aligned}$$

such that

$$||(a_k^N + b_k^N) - 0|| \leq \frac{1}{N} \tag{1}$$

for $k = 2, 3$ and

$$||(a_1^N + b_1^N) - p|| \leq \frac{1}{N} \ . \tag{2}$$

Notice inequality 1 tells us that

$$\lim_{N\to\infty} a_k^N = \lim_{N\to\infty} b_k^N = 0$$

for $k = 2, 3$. We now pick any $x \in X \setminus \{x_0\}$. Then inequality 2 tells us

$$|a_1^N(x) + b_1^N(x)| \leq \frac{1}{N}$$

and we can conclude $\lim_{N\to\infty} a_1^N(x) = \lim_{N\to\infty} b_1^N(x) = 0$. On the point x_0, we can say

$$|a_1^N(x_0) + b_1^N(x_0) - 1| \leq \frac{1}{N}$$

Hence

$$\lim_{N\to\infty} a_1^N(x_0) + b_1^N(x_0) = 1 \ . \tag{3}$$

This tells us $||a^N|| \leq \max\{1/N, a_1^N(x_0)\}$, $||b^N|| \leq \max\{1/N, b_1^N(x_0)\}$. As $||a^N|| - 1| \leq 1/N$ and $|||b_N|| - 1| \leq 1/N$, $\lim_{N\to\infty} a_1^N(x_0) = \lim_{N\to\infty} b_1^N(x_0) = 1$. By equation 3, this is a contradiction. □

As $\phi(p, p, 0) = 0$, we have a contradiction by the spectral theorem. □

Corollary 4.5. *(Improvement to [EagVig]) Given a compact Hausdorff space X with* $\dim(X) = 0$*,* $\mathrm{Th}(C(X))$ *admits quantifier elimination if and only if X has no isolated points.*

Proof. There are two cases. If X has no isolated points, then [EagVig] states that $\mathrm{Th}(C(X))$ admits quantifier elimination. Otherwise, by proposition 4.7, $\mathrm{Th}(C(X))$ does not admit quantifier elimination. □

Remark. *Notice that when we have a unital C^*-algebra A and $a := (a_1, \ldots, a_n) \in A^n$, that* $\mathrm{sp}(a) = \bigcup_{k=1}^n \mathrm{sp}(a_k)$*.*

Proposition 4.8. *Given a unital Abelian C^*-algebra A, the theory* $\mathrm{Th}(M_2(A))$ *does not admit quantifier elimination.*

Proof. Consider the elements $a = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0\right)$ and $b = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$ in $M_n(A)^2$. By the previous remark, $\mathrm{sp}(a) = \mathrm{sp}(b)$. Now, we consider

$$\phi\,(x, y) = \sup_{F,G} \max_{i,j=1,2} \{||(xFxxGx - xGxxFx)_{ij}||, ||(yFyyGy - yGyyFy)_{ij}||\} \ .$$

where x,y,F, and G are treated as 2×2 matrices. Interpreted in a C^*-algebra B, this asserts that the space $(x,y)B^2(x,y)$ is Abelian. Notice that the space $aM_2(A)^2a$ is Abelian since the second component just drops $M_2(A)$ down to zero and the first component creates a projection of $M_2(A)$ onto the first component. However, in $bM_2(A)^2b = M_2(A)$, since $M_2(A)$ is not Abelian, $bM_2(A)^2b$ is not Abelian. Hence $\phi(a)^{M_2(A)} \neq \phi(b)^{M_2(A)}$. By the spectral theorem, this shows us $\mathrm{Th}(M_2(A))$ does not have quantifier elimination. □

Corollary 4.6. *Given a unital Abelian C^*-algebra A, the theory* $\mathrm{Th}\left(\bigcup_{n\in\mathbb{N}} M_n(A)\right)$ *does not admit quantifier elimination.*

Proof. Use $\phi(x,y)$ as in proposition 4.8. The same result still holds replacing $M_2(A)$ with $\bigcup_{n\in\mathbb{N}} M_n(A)$. □

Proposition 4.9. *Let A be a unital Abelian C^*-algebra. Then the theory of* $\mathrm{Th}(M_n(A))$ *does not admit quantifier elimination for all $n \geq 3$.*

Proof. Fix $n \in \mathbb{N}$ such that $n \geq 3$. Let $\varphi(x)$ be the formula

$$\sup_{a,b\,\in\, M_n(A)} (||xax\,xbx - xbx\,xax||).$$

Since $n \geq 3$, we can find matrices p, q in $M_n(A)$, where

$$p = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that $pM_n(A)p \cong M_1(A) = A$, since pMp is only non-zero on $(pMp)_{11}$ for all $M \in M_n(A)$. Similarly, $qM_n(A)q \cong M_2(A)$. Hence we see that $\varphi(p) = 0$ since $pM_n(A)p$ is an Abelian C^*-algebra, while the non-commutativity of $qM_n(A)q$ implies $\varphi(q) > 0$. Thus $\mathrm{Th}(M_n(A))$ cannot have quantifier elimination. □

5 Open Questions

We have classified a large number of Abelian C^*-algebras. Nevertheless, two big questions remain.

- Are there any spaces other than the Cantor space such that its continuous functions admit quantifier elimination?
- Are there any non-abelian C^*-algebras with quantifier elimination?

6 Comments and Acknowledgments

We (authors) present this article as the *Project Report* during our visit at the Fields Institute as part of the **FIELDS UNDERGRADUATE SUMMER RESEARCH PROGRAM 2014**.

We want to thank the Fields Institute for giving us a great opportunity to become involved in mathematical research and for providing financial support as well.

Also we thank our supervisors, Dr. Bradd Hart and Dr. Ilijas Farah, for their attention and helpful comments during the summer.

Special thanks to Christopher Eagle, who helped us *many* times, from solving small doubts to providing us with ideas, counterexamples and results.

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