The Model Theory of C^*-algebras

Diego Caudillo Amador, Jonathan Berger, Jamal Kawach,
Se-jin Kim, Yushen Zhang

August 26, 2014

With thanks to Bradd Hart, Ilijas Farah, and Christopher Eagle

Basics of C^*-algebras

- A C^*-algebra is a subalgebra of the algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H with the operator norm

$$||a|| = \sup_{||x|| \leq 1} ||ax|| \, .$$

Basics of C^*-algebras

- A C^*-algebra is a subalgebra of the algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H with the operator norm

$$||a|| = \sup_{||x|| \leq 1} ||ax|| \; .$$

- *E.g.*, the space $M_n(\mathbb{C})$ of $n \times n$-matrices on $\mathbb{C}^n$ is a C^*-algebra.

Basics of C^*-algebras

- A C^*-algebra is a subalgebra of the algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H with the operator norm

$$||a|| = \sup_{||x|| \leq 1} ||ax|| \,.$$

- *E.g.*, the space $M_n(\mathbb{C})$ of $n \times n$-matrices on $\mathbb{C}^n$ is a C^*-algebra.
- One can abstractly axiomatize these C^*-algebras.

Basics of C^*-algebras

- A C^*-algebra is a subalgebra of the algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H with the operator norm

$$||a|| = \sup_{||x|| \leq 1} ||ax|| .$$

- *E.g.*, the space $M_n(\mathbb{C})$ of $n \times n$-matrices on $\mathbb{C}^n$ is a C^*-algebra.
- One can abstractly axiomatize these C^*-algebras.
- A C^*-algebra is unital if there is a multiplicative identity denoted 1.

Basics of C^*-algebras

- A C^*-algebra is a subalgebra of the algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H with the operator norm

$$||a|| = \sup_{||x|| \leq 1} ||ax|| \; .$$

- *E.g.*, the space $M_n(\mathbb{C})$ of $n \times n$-matrices on $\mathbb{C}^n$ is a C^*-algebra.
- One can abstractly axiomatize these C^*-algebras.
- A C^*-algebra is unital if there is a multiplicative identity denoted 1.
- A C^*-algebra is Abelian if the multiplication operation commutes.

The Gelfand-Naimark theorem

We will only be concerned with the unital Abelian case today. In this case, we have the following theorem:

The Gelfand-Naimark theorem

We will only be concerned with the unital Abelian case today. In this case, we have the following theorem:

Gelfand-Naimark

Given any unital Abelian C^*-algebra A, there is a compact Hausdorff space X such that

$$A \cong C(X)$$

isometrically, where $C(X)$ is the space of continuous functions on X with addition and multiplcation defined pointwise and

$$f^*(x) := \overline{f(x)}\ .$$

The Gelfand-Naimark theorem

We will only be concerned with the unital Abelian case today. In this case, we have the following theorem:

Gelfand-Naimark

Given any unital Abelian C^*-algebra A, there is a compact Hausdorff space X such that

$$A \cong C(X)$$

isometrically, where $C(X)$ is the space of continuous functions on X with addition and multiplcation defined pointwise and

$$f^*(x) := \overline{f(x)}\ .$$

We now turn to continuous logic.

What is continuous logic?

What is continuous logic?

Formulas and things

L formulas

An L-formula is any expression made by combining

Formulas and things

L formulas

An L-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;

Formulas and things

L formulas

An L-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;
2. the operations max, min, and $\dot{-}$, where $x \dot{-} y := \max\{x - y, 0\}$;

Formulas and things

L formulas

An L-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;
2. the operations max, min, and $\dot{-}$, where $x \dot{-} y := \max\{x - y, 0\}$;
3. multiplication of a function by a real number; and

Formulas and things

L formulas

An L-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;
2. the operations max, min, and $\dot{-}$, where $x \dot{-} y := \max\{x - y, 0\}$;
3. multiplication of a function by a real number; and
4. the operations $\sup_{||x|| \leq 1}$ and $\inf_{||x|| \leq 1}$

L formulas

An L-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;
2. the operations max, min, and $\dot{-}$, where $x \dot{-} y := \max\{x - y, 0\}$;
3. multiplication of a function by a real number; and
4. the operations $\sup_{||x|| \leq 1}$ and $\inf_{||x|| \leq 1}$

such that the resulting function *makes sense*.

Formulas and things

L formulas

An L-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;
2. the operations max, min, and $\dot{-}$, where $x \dot{-} y := \max\{x - y, 0\}$;
3. multiplication of a function by a real number; and
4. the operations $\sup_{||x|| \leq 1}$ and $\inf_{||x|| \leq 1}$

such that the resulting function *makes sense*.

- When we plug in elements of our C^*-algebra into the free variables, we can evaluate a formula to a real number.

Formulas and things

L formulas

An L-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;
2. the operations max, min, and $\dot{-}$, where $x \dot{-} y := \max\{x - y, 0\}$;
3. multiplication of a function by a real number; and
4. the operations $\sup_{||x|| \leq 1}$ and $\inf_{||x|| \leq 1}$

such that the resulting function *makes sense*.

- When we plug in elements of our C^*-algebra into the free variables, we can evaluate a formula to a real number.
- We call any combination of max, min, $\dot{-}$, and multiplication by real numbers connectives

Formulas and things

L formulas

An L-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;
2. the operations max, min, and $\dot{-}$, where $x \dot{-} y := \max\{x - y, 0\}$;
3. multiplication of a function by a real number; and
4. the operations $\sup_{||x||\leq 1}$ and $\inf_{||x||\leq 1}$

such that the resulting function *makes sense*.

- When we plug in elements of our C^*-algebra into the free variables, we can evaluate a formula to a real number.
- We call any combination of max, min, $\dot{-}$, and multiplication by real numbers connectives
- We call $\sup_{||x||\leq 1}$ and $\inf_{||x||\leq 1}$ quantifiers.

Formulas and things

L formulas

An *L*-formula is any expression made by combining

1. $||p(\bar{x})||$ for $p(\bar{x})$ a *-polynomial;
2. the operations max, min, and $\dot{-}$, where $x \dot{-} y := \max\{x - y, 0\}$;
3. multiplication of a function by a real number; and
4. the operations $\sup_{||x|| \leq 1}$ and $\inf_{||x|| \leq 1}$

such that the resulting function *makes sense*.

- When we plug in elements of our C^*-algebra into the free variables, we can evaluate a formula to a real number.
- We call any combination of max, min, $\dot{-}$, and multiplication by real numbers connectives
- We call $\sup_{||x|| \leq 1}$ and $\inf_{||x|| \leq 1}$ quantifiers.
- We call all formulas with no free variables sentences.

The continuous logic

- max acts like $\wedge$

The continuous logic

- max acts like $\wedge$
- min acts like $\vee$

The continuous logic

- max acts like $\wedge$
- min acts like $\vee$
- $\dot{-}$ acts like $\rightarrow$

The continuous logic

- max acts like $\wedge$
- min acts like $\vee$
- $\dot{-}$ acts like $\rightarrow$
- $\sup_{||x|| \leq 1}$ acts like $\forall x$

The continuous logic

- max acts like $\wedge$
- min acts like $\vee$
- $\dot{-}$ acts like $\rightarrow$
- $\sup_{||x||\leq 1}$ acts like $\forall x$
- $\inf_{||x||\leq 1}$ acts like $\exists x$

The continuous logic

- max acts like $\wedge$
- min acts like $\vee$
- $\dot{-}$ acts like $\rightarrow$
- $\sup_{||x|| \leq 1}$ acts like $\forall x$
- $\inf_{||x|| \leq 1}$ acts like $\exists x$
- Notice we never referred to the specific C^*-algebra in question.

What is model theory?

- Given two C^*-algebras A and B, we can ask when they have the same value on sentences.

What is model theory?

- Given two C^*-algebras A and B, we can ask when they have the same value on sentences.
- If they have the same value for enough sentences, then it is possible to solve a problem about A by solving it for B!

A calculation

A simple exercise

Calculate

$$\sup_{||x||\leq 1} \inf_{||y||\leq 1} \sup_{||z||\leq 1} \max\{||x^2 - y + z - xyz + x - xy - 2||, \\ \min\{||x^6 - y^{90200} + z^{299792458} - 56834||, ||1 - y^{902}x^{808}||\}\}$$

interpreting the symbols in $C[0,1]$.

Huh?

This is a hard calculation.

Huh?

This is a hard calculation.

Quantifier Elimination

A admits quantifier elimination provided that, for any L formula $\varphi(x_1, \ldots, x_n)$, there exists a sequence $\psi_N(x_1, \ldots, x_n)$ of formulas without any instance of quantifiers such that

$$\lim_{N\to\infty} \sup_{x_1,\ldots x_n \in D_1} |\psi_N(x_1, \ldots, x_n) - \varphi(x_1, \ldots, x_n)| = 0$$

where the formulas are interpreted in the C^*-algebra A.

The spectrum

- Fix a C^*-algebra A.

The spectrum

- Fix a C^*-algebra A.
- Define the spectrum of $a \in A$ as

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}$$

The spectrum

- Fix a C^*-algebra A.
- Define the spectrum of $a \in A$ as

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}$$

- These generalize the idea of eigenvalues to any space.

The spectrum

- Fix a C^*-algebra A.
- Define the spectrum of $a \in A$ as

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}$$

- These generalize the idea of eigenvalues to any space.
- The spectrum $\mathrm{sp}(a)$ is a non-empty compact set.

The spectrum

- Fix a C^*-algebra A.
- Define the spectrum of $a \in A$ as

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}$$

- These generalize the idea of eigenvalues to any space.
- The spectrum $\mathrm{sp}(a)$ is a non-empty compact set.
- In the case when $A = C(X)$, $\mathrm{sp}(a) = \mathrm{range}(a)$.

The spectral theorem

- The spectral theorem tells us

The spectral theorem

- The spectral theorem tells us

Spectral theorem

Given a normal operator a in a C^* algebra A, there is an isometry

$$u : C^*(1, a) \to C(\mathrm{sp}(a))$$

where $C^*(1, a)$ is the C^*-algebra generated by 1 and a, $u(1) = 1$, and $u(a)$ is the linear function $x \mapsto x$.

The spectral theorem

- The spectral theorem tells us

Spectral theorem

Given a normal operator a in a C^* algebra A, there is an isometry

$$u : C^*(1, a) \to C(\mathrm{sp}(a))$$

where $C^*(1, a)$ is the C^*-algebra generated by 1 and a, $u(1) = 1$, and $u(a)$ is the linear function $x \mapsto x$.

- Let $a, b \in C(X)$ have $\mathrm{sp}(a) = \mathrm{sp}(b)$.

The spectral theorem

- The spectral theorem tells us

Spectral theorem

Given a normal operator a in a C^* algebra A, there is an isometry

$$u : C^*(1, a) \to C(\mathrm{sp}(a))$$

where $C^*(1, a)$ is the C^*-algebra generated by 1 and a, $u(1) = 1$, and $u(a)$ is the linear function $x \mapsto x$.

- Let $a, b \in C(X)$ have $\mathrm{sp}(a) = \mathrm{sp}(b)$.
- The spectral theorem guarantees that there is an isometry

$$C^*(1, a) \cong C^*(1, b)$$

given by sending 1 to 1 and a to b.

- Given a formula $\varphi(x)$ with no quantifiers, $\varphi(x) = u(||p_1(x)||, \ldots, ||p_n(x)||)$ for some *-polynomials $p_1, \ldots, p_n$ and u some connective.

- Given a formula $\varphi(x)$ with no quantifiers, $\varphi(x) = u(||p_1(x)||, \dots, ||p_n(x)||)$ for some *-polynomials $p_1, \dots, p_n$ and u some connective.
- Since $\mathrm{sp}(a) = \mathrm{sp}(b)$, $||p_k(a)|| = ||p_k(b)||$.

- Given a formula $\varphi(x)$ with no quantifiers, $\varphi(x) = u(||p_1(x)||, \ldots, ||p_n(x)||)$ for some *-polynomials $p_1, \ldots, p_n$ and u some connective.
- Since $\mathrm{sp}(a) = \mathrm{sp}(b)$, $||p_k(a)|| = ||p_k(b)||$.
- Therefore $\varphi(a) = \varphi(b)$.

An illustrative example

The model $C[0,1]$ does not eliminate quantifiers.

An illustrative example

The model $C[0,1]$ does not eliminate quantifiers.

- Getting quantifier elimination is not going to be easy!

Some things do admit Q.E.

There are a few cases in which quantifier elimination holds.

Some things do admit Q.E.

There are a few cases in which quantifier elimination holds.

Eagle, Vignati

Given a compact Hausdorff space X in which

Some things do admit Q.E.

There are a few cases in which quantifier elimination holds.

Eagle, Vignati

Given a compact Hausdorff space X in which

- X is of dimension 0, and

Some things do admit Q.E.

There are a few cases in which quantifier elimination holds.

Eagle, Vignati

Given a compact Hausdorff space X in which

- X is of dimension 0, and
- X has no isolated points, then

Some things do admit Q.E.

There are a few cases in which quantifier elimination holds.

Eagle, Vignati

Given a compact Hausdorff space X in which

- X is of dimension 0, and
- X has no isolated points, then

the space $C(X)$ has quantifier elimination.

Some things do admit Q.E.

There are a few cases in which quantifier elimination holds.

Eagle, Vignati

Given a compact Hausdorff space X in which

- X is of dimension 0, and
- X has no isolated points, then

the space $C(X)$ has quantifier elimination.

- For example, given the Cantor space $2^{\mathbb{N}}$, $C(2^{\mathbb{N}})$ has quantifier elimination.

Some things do admit Q.E.

There are a few cases in which quantifier elimination holds.

Eagle, Vignati

Given a compact Hausdorff space X in which

- X is of dimension 0, and
- X has no isolated points, then

the space $C(X)$ has quantifier elimination.

- For example, given the Cantor space $2^{\mathbb{N}}$, $C(2^{\mathbb{N}})$ has quantifier elimination.
- However, simple spaces like $\mathbb{C}^n$ does not admit quantifier elimination.

Improving the result

We cannot do better than this.

Improving the result

We cannot do better than this.

No isolated point

Given any space X with an isolated point, $C(X)$ does not admit quantifer elimination.

A criterion for failing to have quantifier elimination

Defintion: Peak property

We say that a function $f : U \to [0, \infty)$ on a compact Hausdorff space U is a peak function provided

A criterion for failing to have quantifier elimination

Defintion: Peak property

We say that a function $f : U \to [0, \infty)$ on a compact Hausdorff space U is a peak function provided

- $\mathrm{sp}(f) = [0, 1]$ and

A criterion for failing to have quantifier elimination

Defintion: Peak property

We say that a function $f : U \to [0, \infty)$ on a compact Hausdorff space U is a peak function provided

- $\mathrm{sp}(f) = [0, 1]$ and
- the set $\{x \in U : f(x) > 1 - \frac{1}{5}\}$ is connected.

A criterion for failing to have quantifier elimination

Defintion: Peak property

We say that a function $f : U \to [0, \infty)$ on a compact Hausdorff space U is a peak function provided

- $\mathrm{sp}(f) = [0, 1]$ and
- the set $\{x \in U : f(x) > 1 - \frac{1}{5}\}$ is connected.

Main result

If U is a compact Hausdorff space with a peak function then $C(U)$ does not admit quantifier elimination.

What spaces satisfy the criterion?

Most nice spaces satisfy our criterion. For example

What spaces satisfy the criterion?

Most nice spaces satisfy our criterion. For example

- n-manifolds satisfy the criterion

What spaces satisfy the criterion?

Most nice spaces satisfy our criterion. For example

- n-manifolds satisfy the criterion
- simplical complexes satisfy the criterion

What spaces satisfy the criterion?

Most nice spaces satisfy our criterion. For example

- *n*-manifolds satisfy the criterion
- simplical complexes satisfy the criterion
- CW-complexes satisfy the criterion

What spaces satisfy the criterion?

Most nice spaces satisfy our criterion. For example

- n-manifolds satisfy the criterion
- simplical complexes satisfy the criterion
- CW-complexes satisfy the criterion
- the Hawaiian earring satisfies the criterion

What spaces satisfy the criterion?

Most nice spaces satisfy our criterion. For example

- *n*-manifolds satisfy the criterion
- simplical complexes satisfy the criterion
- CW-complexes satisfy the criterion
- the Hawaiian earring satisfies the criterion
- etc.

This isn't looking good for quantifier elimination

We have even more negative results:

This isn't looking good for quantifier elimination

We have even more negative results:

Thick spaces don't have quantifier elimination

If X is a path-connected, compact, Hausdorff space then $C([0,1] \times X)$ does not have quantifer elimination.

This isn't looking good for quantifier elimination

We have even more negative results:

Thick spaces don't have quantifier elimination

If X is a path-connected, compact, Hausdorff space then $C([0,1] \times X)$ does not have quantifer elimination.

E.g., for the Hilbert cube $[0,1]^{\mathbb{N}}$, $C([0,1]^{\mathbb{N}})$ does not have quantifier elimination.

Where to go from here

We have classified a lot of spaces. This leaves us with

Where to go from here

We have classified a lot of spaces. This leaves us with

Question

Are there any spaces other than $C(2^{\mathbb{N}})$ which admits quantifier elimination?

Where to go from here

We have classified a lot of spaces. This leaves us with

Question

Are there any spaces other than $C(2^{\mathbb{N}})$ which admits quantifier elimination?

- Here is an example of a space which is not classified: does $C(2^{\mathbb{N}} \times [0,1])$ admit quantifier elimination?

Where to go from here

We have classified a lot of spaces. This leaves us with

Question

Are there any spaces other than $C(2^{\mathbb{N}})$ which admits quantifier elimination?

- Here is an example of a space which is not classified: does $C(2^{\mathbb{N}} \times [0,1])$ admit quantifier elimination?
  - Actually, yesterday we concluded $C(2^{\mathbb{N}} \times [0,1])$ does not have quantifier elimination.

Where to go from here

We have classified a lot of spaces. This leaves us with

Question

Are there any spaces other than $C(2^{\mathbb{N}})$ which admits quantifier elimination?

- Here is an example of a space which is not classified: does $C(2^{\mathbb{N}} \times [0,1])$ admit quantifier elimination?
  - Actually, yesterday we concluded $C(2^{\mathbb{N}} \times [0,1])$ does not have quantifier elimination.
- What about non-Abelian C^*-algebras?

Where to go from here

We have classified a lot of spaces. This leaves us with

Question

Are there any spaces other than $C(2^{\mathbb{N}})$ which admits quantifier elimination?

- Here is an example of a space which is not classified: does $C(2^{\mathbb{N}} \times [0,1])$ admit quantifier elimination?
  - Actually, yesterday we concluded $C(2^{\mathbb{N}} \times [0,1])$ does not have quantifier elimination.
- What about non-Abelian C^*-algebras?
- We can show that $M_n(C(X))$ for $n \geq 2$ does not admit quantifier elimination, but the general question is still open.