On the Hardy-Sobolev operator with a boundary singularity

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(Joint work with Frédéric Robert (Université de Lorraine)

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Given a smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, find a metric conformal to g with constant scalar curvature.
It amounts to finding a positive solution for

$$-\frac{4(n-1)}{n-2}\Delta u + \lambda u = u^{2^*-1} \quad \text{on M}, \tag{1}$$

or to minimize

$$\mu(M) = \inf \left\{ \frac{\int_M (\frac{4(n-1)}{n-2}|\nabla u|^2 + \lambda |u|^2)\, dV_g}{\left(\int_M |u|^{2^*}\, dV_g\right)^{\frac{2}{2^*}}} ; u \in D^{1,2}(M), u \neq 0 \right\},$$

where λ is the scalar curvature with respect to g.
(Yamabe, Trudinger, Aubin). The Yamabe problem can be solved on any compact manifold M with $\mu(M) < \mu(\mathbb{S}^n)$, where $\mathbb{S}^n$ is the sphere with its standard metric.

(Aubin). If M has dimension $n \geq 6$ and is not locally conformally flat then $\mu(M) < \mu(\mathbb{S}^n)$.

(Schoen). If M has dimension 3, 4, or 5, or if M is locally conformally flat, then $\mu(M) < \mu(\mathbb{S}^n)$ unless M is conformal to the standard sphere.

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What happens if $M \subset \mathbb{R}^n$. Can one still solve (1) with Dirichlet boundary conditions–say?

Now assume $\Omega \subset \mathbb{R}^n$. Then,

$$-\Delta u + \lambda u = u^{2^*-1} \quad \text{on } \Omega, \tag{2}$$

has no solution if $\lambda \geq 0$.
The best constant in the Sobolev inequality

$$\mu(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2)\, dx}{\left(\int_\Omega |u|^{2^*}\, dx\right)^{\frac{2}{2^*}}}; u \in D^{1,2}(\Omega), u \neq 0 \right\},$$

is never attained unless Ω is essentially $\mathbb{R}^n$. Actually, $\mu(\Omega) = \mu(\mathbb{R}^n)$ for every $\Omega \subset \mathbb{R}^n$.
Three ways to break the homogeneity of the problem:

1. Brezis-Nirenberg (1983) $-\Delta u + \lambda u = u^{2^*-1}$ has a positive solution if $-\lambda_1(\Omega) < \lambda < 0$ and $n \geq 4$. Dimension $n = 3$ is different!: Druet.
2. Bahri-Coron (1987) $-\Delta u = u^{2^*-1}$ has a positive solution, if Ω is an annular domain (or if $H_d(\Omega, \mathbb{Z}_2) \neq 0$ for some $d > 0$, e.g., Ω non-contractible in $\mathbb{R}^3$.)
3. Ghoussoub-Kang (2003) Singularize the problem!!!

Classical inequalities on $\mathbb{R}^n$, $n \geq 3$,

Hardy's inequality:

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^n}|\nabla u|^2\,dx \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n).$$

Sobolev inequality:

$$\left(\int_{\mathbb{R}^n}|u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \leq C(n)\int_{\mathbb{R}^n}|\nabla u|^2\,dx \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n).$$

Hardy-Sobolev inequality: For $s \in [0,2]$, $2^\star(s) := \frac{2(n-s)}{n-2}$.

$$\left(\int_{\mathbb{R}^n}\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \leq C(n,s)\int_{\mathbb{R}^n}|\nabla u|^2\,dx \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n).$$

Caffarelli-Kohn-Nirenberg: For $a \leq b \leq b+1$, $a < \frac{n-2}{2}$, and $p := \frac{2n}{n-2+2(b-a)}$,

$$\left(\int_{\mathbb{R}^n}|x|^{-bp}|u|^p\,dx\right)^{\frac{2}{p}} \leq C(a,b,n)\int_{\mathbb{R}^n}|x|^{-2a}|\nabla u|^2\,dx \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n).$$

Writing $v(x) := |x|^{-a}u(x)$, this rewrites with $\gamma := a(n-2-a) < \frac{(n-2)^2}{4}$ as:

$$\left(\int_{\mathbb{R}^n}\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \leq C(n,\gamma,s)\int_{\mathbb{R}^n}\left(|\nabla u|^2-\gamma\frac{u^2}{|x|^2}\right)\,dx \quad \text{for } u \in C_c^\infty(\mathbb{R}^n).$$

Define for any $\Omega \subset \mathbb{R}^n$, the best constant

$$\mu_{\gamma,s}(\Omega) := \inf_{u \in D^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2}\right) dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s} dx\right)^{\frac{2}{2^\star(s)}}},$$

Again, if the singularity $0 \in \Omega$, then for $0 \leq s < 2$ and $\gamma < (n-2)^2/4$,

$$\mu_{\gamma,s}(\Omega) = \mu_{\gamma,s}(\mathbb{R}^n).$$

The infimum is never attained unless $\Omega = \mathbb{R}^n$.

What about domains such that $0 \in \partial\Omega$?
Are there extremals for $\mu_{\gamma,s}(\Omega)$? i.e., positive solutions to the Euler-Lagrange equation

$$\begin{cases} -\Delta u - \gamma \frac{u}{|x|^2} = \frac{u^{2^\star(s)-1}}{|x|^s} & \text{on } \Omega \\ u > 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

- Gh-Robert (2006) If $\gamma = 0$ and $s > 0$, then there are extremals for all $n \geq 3$, provided the mean curvature of $\partial\Omega$ at 0 is negative. Hence, there are positive solutions for

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Same for $s = 0$ provided $n \geq 4$ and $\gamma > 0$.
What happens if:

1. $s = 0$ and $n = 3$.
2. $\gamma \geq \frac{(n-2)^2}{4}$.

Best constants in Hardy's inequality?

Consider first the best constant in the Hardy inequality

$$\gamma_H(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega \frac{u^2}{|x|^2} \, dx};\ u \in D^{1,2}(\Omega) \setminus \{0\} \right\} \quad D^{1,2}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|},\ \|u\| := \|\nabla u\|_2.$$

Easy to see that if $0 \in \Omega$, then $\gamma_H(\Omega)$ does not depend on the domain $\Omega \subset \mathbb{R}^n$

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HOWEVER,
Proposition: For $1 \le k \le n$, we have:

$$\left(\frac{n+2k-2}{2}\right)^2 = \inf_u \frac{\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k}} |\nabla u|^2\,dx}{\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k}} \frac{u^2}{|x|^2}\,dx},$$

where the infimum is taken on $u \in D^{1,2}(\mathbb{R}_+^k \times \mathbb{R}^{n-k})\setminus\{0\}$ is never achieved. In particular,

$$\gamma_H(\mathbb{R}_+^n) = \frac{n^2}{4}.$$

Hardy best constants when $0 \in \partial\Omega$

Unlike the case when 0 is in the interior of a domain, we have the following

Proposition: If $0 \in \partial\Omega$, then

1. $\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \frac{n^2}{4}$.
2. $\gamma_H(\Omega) = \frac{n^2}{4}$ for every Ω such that $0 \in \partial\Omega$ and $\Omega \subset \mathbb{R}^n_+$.
3. $\inf\{\gamma_H(\Omega);\, 0 \in \partial\Omega\} = \frac{(n-2)^2}{4}$.
4. For every $\epsilon > 0$, there exists a smooth domain Ω_ϵ such that $0 \in \partial\Omega_\epsilon$, $\mathbb{R}^n_+ \subsetneq \Omega_\epsilon \subsetneq \mathbb{R}^n$ and $\frac{n^2}{4} - \epsilon \leq \gamma_H(\Omega_\epsilon) < \frac{n^2}{4}$.

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3. $\inf\{\gamma_H(\Omega);\ 0 \in \partial\Omega\} = \frac{(n-2)^2}{4}$.
4. For every $\epsilon > 0$, there exists a smooth domain Ω_ϵ such that $0 \in \partial\Omega_\epsilon$, $\mathbb{R}^n_+ \subsetneq \Omega_\epsilon \subsetneq \mathbb{R}^n$ and $\frac{n^2}{4} - \epsilon \leq \gamma_H(\Omega_\epsilon) < \frac{n^2}{4}$.

.... and a Caffarelli-Kohn-Nirenberg inequality on $\mathbb{R}^n_+$:
There exists $C := C(a,b,n) > 0$ such that for $u \in C_c^\infty(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$,

$$\left(\int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} |x|^{-bq} \left(\Pi_{i=1}^k x_i\right)^q |u|^q\right)^{\frac{2}{q}} \leq C \int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} \left(\Pi_{i=1}^k x_i\right)^2 |x|^{-2a} |\nabla u|^2 dx,$$

where

$$-\infty < a < \frac{n-2+2k}{2}, \quad 0 \leq b-a \leq 1, \quad q = \frac{2n}{n-2+2(b-a)}. \tag{6}$$

Back to the best constant

More importantly, we then have for any $\gamma < n^2/4$, $0 \le s \le 2$, $2^\star(s) := \frac{2(n-s)}{n-2}$,

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}} \le C''_{n,\gamma,s} \int_{\mathbb{R}^n_+} \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2}\right) dx \quad \text{for } u \in C_c^\infty(\mathbb{R}^n).$$

For $\Omega \subset \mathbb{R}^n$, the best constant $\mu_{\gamma,s}(\Omega) := \inf\{I_{s,\gamma}(\Omega); u \in C_c^\infty(\Omega) \setminus \{0\}\}$, where

$$I_{s,\gamma}(u) := \frac{\int_\Omega \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2}\right) dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}}},$$

Again, for any Ω with $0 \in \Omega$, we have for $0 \le s < 2$ and $\gamma < \gamma_H(\Omega) = (n-2)^2/4$

$$\mu_{\gamma,s}(\Omega) = \mu_{\gamma,s}(\mathbb{R}^n).$$

The infimum is never attained unless $\Omega = \mathbb{R}^n$.

What about domains such that $0 \in \partial\Omega$?

We already know that

- $\mu_{\gamma,s}(\Omega) > 0$, whenever $0 \le s < 2$ and $\gamma < \gamma_H(\Omega) < n^2/4$.
- $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+)$, hence is attained if $s > 0$, $n \ge 3$ and $\gamma < \frac{(n-2)^2}{4}$.

What happens in the remaining cases? that is when

$$\gamma \in \left[\frac{(n-2)^2}{4}, \gamma_H(\Omega)\right) \subset \left[\frac{(n-2)^2}{4}, \frac{n^2}{4}\right)$$

Theorem

Let Ω be a bounded smooth domain of $\mathbb{R}^n$ ($n \geq 3$) such that $0 \in \partial\Omega$. In particular $\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \frac{n^2}{4}$. Let $0 \leq s < 2$.

1. *If $\gamma_H(\Omega) \leq \gamma < \frac{n^2}{4}$, then there are extremals for $\mu_{\gamma,s}(\Omega)$ for every $s \in [0,2)$ and any $n \geq 3$.*
2. *If $\gamma < \gamma_H(\Omega)$ and $s > 0$, then*
   - *$\gamma \leq \frac{n^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
   - *$\gamma > \frac{n^2-1}{4}$ and the Hardy b-mass $m_\gamma(\Omega)$ is positive.*

Table: Singular Sobolev-Critical term: $s > 0$

Hardy term	**Dimension**	**Geometric condition**	**Extremal**
$-\infty < \gamma \leq \frac{n^2-1}{4}$	$n \geq 3$	Negative mean curvature at 0	Yes
$\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$	$n \geq 3$	Positive boundary-mass	Yes

Critical cases: $s = 0$

Table: Non-singular Sobolev-Critical term: $s = 0$

Hardy term	Dim.	Geometric condition	Extremal
$0 < \gamma \leq \frac{n^2-1}{4}$	$n = 3$ $n \geq 4$	Negative mean curvature at 0 & Positive internal mass Negative mean curvature at 0	Yes Yes
$\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$	$n = 3$ $n \geq 4$	Positive boundary-mass & Positive internal mass Positive boundary mass	Yes Yes
$\gamma \leq 0$	$n \geq 3$	–	No

Theorem Let Ω be a bounded smooth domain of $\mathbb{R}^3$ such that $0 \in \partial\Omega$. In particular $\frac{1}{4} < \gamma_H(\Omega) \leq \frac{9}{4}$.

- If $\gamma_H(\Omega) \leq \gamma < \frac{9}{4}$, then there are extremals for $\mu_{\gamma,0}(\Omega)$.
- If $0 < \gamma < \gamma_H(\Omega)$, and if there exists $x_0 \in \Omega$ such that $R_\gamma(\Omega, x_0) > 0$, then there are extremals for $\mu_{\gamma,0}(\Omega)$, under either one of the following conditions:
  1. $\gamma \leq 2$ and the mean curvature of $\partial\Omega$ at 0 is negative.
  2. $\gamma > 2$ and the mass $m_\gamma(\Omega)$ is positive.

Standard scheme but the challenge is in the implementation

Standard fact: (Dating back to the Yamabe problem (Trudinger, Aubin)

IF $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+)$, then there are extremals for $\mu_{\gamma,s}(\Omega)$.

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Another standard fact: Use the nonnegative extremal $U \in D^{1,2}(\mathbb{R}^n_+)$ for $\mu_{\gamma,s}(\mathbb{R}^n_+)$ (if it exists) to build test functions. Besides existence, one needs information on the profile i.e. behavior at 0 and at infinity.

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Proposition (Existence on $\mathbb{R}^n_+$): Fix $\gamma < \frac{n^2}{4}$ and $s \in [0,2)$ with $n \geq 3$. Then,

1. If either $\{s > 0\}$ or $\{s = 0,\ \gamma > 0 \text{ and } n \geq 4\}$, then $\mu_{\gamma,s}(\mathbb{R}^n_+)$ is attained.
2. If $\{s = 0 \text{ and } \gamma \leq 0\}$, there are no extremals for $\mu_{\gamma,0}(\mathbb{R}^n_+)$.
3. The only unknown situation on $\mathbb{R}^n_+$ is when $\{s = 0,\ n = 3 \text{ and } \gamma > 0\}$, BUT:

   If $\mu_{\gamma,0}(\mathbb{R}^n_+)$ is not attained, then $\mu_{\gamma,0}(\mathbb{R}^n_+) = \inf_{u \in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\star}\, dx\right)^{\frac{2}{2^\star}}}$.

Theorem (Symmetry on $\mathbb{R}^n_+$): If $u \in D^{1,2}(\mathbb{R}^n_+)$ is an extremal for $\mu_{\gamma,0}(\mathbb{R}^n_+)$, then $u \circ \sigma = u$ for all isometry of $\mathbb{R}^n$ such that $\sigma(\mathbb{R}^n_+) = \mathbb{R}^n_+$. In particular, there exists $v \in C^\infty((0,+\infty) \times \mathbb{R})$ such that for all $x_1 > 0$ and all $x' \in \mathbb{R}^{n-1}$, we have that $u(x_1, x') = v(x_1, |x'|)$.

Question (Behavior at 0 and ∞)

Define $u_\epsilon(x) := \eta(x)\left(\epsilon^{-\frac{n-2}{2}} U(\epsilon^{-1}\cdot)\right) \circ \varphi^{-1}(x)$, where η is a cut-off around 0 and φ is a chart mapping locally $\mathbb{R}^n_+$ on Ω.

1. If $\gamma < \frac{n^2-1}{4}$, then $\int_{\partial\mathbb{R}^n_+} |x|^2|\nabla U|^2\, dx < +\infty$, then the test functions u_ϵ work:

$$\boxed{I_{s,\gamma}(u_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+) + C_{n,s,\gamma}\cdot H(0)\cdot\epsilon + o(\epsilon)}$$

   where $H(0)$ is the mean curvature of $\partial\Omega$ at 0.

The three main cases

Define $u_\epsilon(x) := \eta(x)\left(\epsilon^{-\frac{n-2}{2}}U(\epsilon^{-1}\cdot)\right)\circ\varphi^{-1}(x)$, where η is a cut-off around 0 and φ is a chart mapping locally $\mathbb{R}^n_+$ on Ω.

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where $H(0)$ is the mean curvature of $\partial\Omega$ at 0.
2. For $\gamma = \frac{n^2-1}{4}$ one needs a finer analysis of the linear operator $L_\gamma := -\Delta - \frac{\gamma}{|x|^2}$ to establish that U behaves exactly like $x_1|x|^{-\alpha_+}$ at infinity.
$$I_{s,\gamma}(u_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+) + C_{n,s,\gamma}\cdot H(0)\cdot\epsilon\ln(1/\epsilon) + o(\epsilon\ln(1/\epsilon))$$

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3. For $\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$, then one constructs global profiles v_ϵ,
$$v_\epsilon(x) := u_\epsilon(x) + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \beta(x), \quad \text{where}$$
$$\beta(x) = m_\gamma(\Omega) \frac{d(x, \partial\Omega)}{|x|^{\alpha_-}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-}}\right) \quad \text{as } x \to 0.$$
in such a way that
$$\boxed{I_{s,\gamma}(v_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+) - C_{n,s,\gamma} \cdot m_\gamma(\Omega) \cdot \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-})}$$
where $m_\gamma(\Omega)$ is the Hardy-singular boundary mass of Ω.

On $\mathbb{R}^n_+$, we define $u_\alpha(x) := x_1|x|^{-\alpha}$ for $\alpha \in \mathbb{R}$. The first remark is that

$$\Delta u_\alpha - \frac{\gamma}{|x|^2} u_\alpha = 0 \text{ in } \mathbb{R}^n_+ \Leftrightarrow \{\alpha = \alpha_-(\gamma) \text{ or } \alpha = \alpha_+(\gamma)\}$$

where

$$\alpha_-(\gamma) := \frac{n}{2} - \sqrt{\frac{n^2}{4} - \gamma} \text{ and } \alpha_+(\gamma) := \frac{n}{2} + \sqrt{\frac{n^2}{4} - \gamma}$$

Note: $\alpha_- < \frac{n}{2} < \alpha_+$, which points to the difference between the two canonical solutions, one is variational namely $x \mapsto x_1|x|^{-\alpha_-(\gamma)}$ is locally in $D^{1,2}(\mathbb{R}^n_+)$, and the "large one" $x \mapsto x_1|x|^{-\alpha_+(\gamma)}$ is not.

Note: the analogy with the case of harmonic functions on $\mathbb{R}^n$ (i.e., solutions of $\Delta u = 0$ on $\mathbb{R}^n \setminus \{0\}$):

$$\Delta |x|^{-\beta} = 0 \text{ in } \mathbb{R}^n \setminus \{0\} \Leftrightarrow \{\beta = 0 \text{ or } \beta = n-2\}.$$

A classification of all solutions on $\mathbb{R}^n_+$

All non-negative solutions of $L_\gamma u = 0$ on $\mathbb{R}^n_+$ turned out to be a linear combination of the two basic ones. Indeed, we prove the following:

Theorem: Let $u \in C^2(\overline{\mathbb{R}^n_+} \setminus \{0\})$ be a nonnegative function such that

$$-\Delta u - \frac{\gamma}{|x|^2} u = 0 \text{ in } \mathbb{R}^n_+ \ ; \ u = 0 \text{ on } \partial\mathbb{R}^n_+.$$

Then there exist $\lambda_-, \lambda_+ \geq 0$ such that

$$u(x) = \lambda_- x_1 |x|^{-\alpha_-} + \lambda_+ x_1 |x|^{-\alpha_+} \text{ for all } x \in \mathbb{R}^n_+.$$

Remark: We eventually show that $x \mapsto d(x, \partial\Omega)|x|^{-\alpha_-(\gamma)}$ is essentially the profile at 0 of any variational solution –positive or not– of equations of the form $L_\gamma u = f(x, u)$, as long as the nonlinearity f is dominated by $C(|v| + \frac{|v|^{2^*(s)}}{|x|^s})$.

Regularity and Hopf-type lemma

Theorem: Consider $u \in D^{1,2}(\Omega)$ that is locally (around 0) a solution to

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x,u)$$

where $|f(x,u)| \leq C|u| \left(1 + \frac{|u|^{2^\star(s)-2}}{|x|^s}\right)$, $\tau > 0$. Then there exists $K \in \mathbb{R}$ such that

$$u(x) = K \frac{d(x,\partial\Omega)}{|x|^{\alpha_-}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-}}\right) \text{ when } x \to 0.$$

Moreover, if $u \geq 0$, $u \not\equiv 0$, then $K > 0$.

Remark 1: when $\gamma = 0$, we have $\alpha_- = 0$ and this is exactly Hopf's lemma ($K = -\partial_\nu u(0) > 0$).

Remark 2: Unlike the case when $L_\gamma = L_0 = -\Delta$) or when the singularity 0 is in the interior, the standard DeGiorgi-Nash-Moser iterative scheme is not sufficient to obtain the required regularity. It only yields that $u \in L^p$ for all $p < p_0 < \frac{n}{\alpha_-(\gamma)-1}$.

Remark 3: However, the improved order p_0 is enough to allow for the inclusion of the nonlinearity $f(x,u)$ in the linear term, hence reducing the analysis to when $f(x,u) \equiv 0$. We get the conclusion by constructing super- and sub- solutions to the linear equation behaving like the canonical solutions.

Profile of the limiting solution

Theorem: Assume $\gamma < \frac{n^2}{4}$ and let $u \in D^{1,2}(\mathbb{R}^n_+)$, $u \geq 0$, $u \not\equiv 0$ be a weak solution to

$$-\Delta u - \frac{\gamma}{|x|^2} u = \frac{u^{2^\star(s)-1}}{|x|^s} \text{ in } \mathbb{R}^n_+.$$

Then, there exist $K_1, K_2 > 0$ such that

$$u(x) \sim_{x\to 0} K_1 \frac{x_1}{|x|^{\alpha_-(\gamma)}} \quad \text{and} \quad u(x) \sim_{|x|\to+\infty} K_2 \frac{x_1}{|x|^{\alpha_+(\gamma)}}.$$

Remark: This description of the profile of variational solutions allows to construct sharper test functions and to prove existence of extremals up to $\gamma = \frac{n^2-1}{4}$. Indeed, the estimates

$$u(x) \leq C x_1 |x|^{-\alpha_+(\gamma)} \quad \text{and} \quad |\nabla u(x)| \leq C|x|^{-\alpha_+(\gamma)} \text{ for all } x \in \mathbb{R}^n_+. \tag{7}$$

and the fact that

$$\gamma < \frac{n^2-1}{4} \Leftrightarrow \alpha_+(\gamma) - \alpha_-(\gamma) > 1$$

yield that if $\gamma < \frac{n^2-1}{4}$, then $|x'|^2|\partial_1 u|^2 = O(|x'|^{2-2\alpha_+(\gamma)})$ as $|x'| \to +\infty$ on $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, from which we could deduce that $x' \mapsto |x'|^2|\partial_1 u(x')|^2$ is in $L^1(\partial\mathbb{R}^n_+)$.
This estimate does not hold when $\gamma \geq \frac{n^2-1}{4}$.

Classification of positive singular solutions

To deal with the remaining cases for γ, we need the following result which describes the general profile of any positive solution of $L_\gamma u = a(x)u$, albeit variational or not.

Theorem: Let $u \in C^2(\overline{\Omega} \cap B_\delta(0) \setminus \{0\})$ be a **positive** solution to

$$\Delta u - \frac{\gamma}{|x|^2} u = 0 \text{ in } \Omega \cap B_\delta(0) \setminus \{0\}; \quad u = 0 \text{ on } (\partial\Omega) \cap B_\delta(0).$$

Then, either u behaves like $d(\cdot, \partial\Omega)|x|^{-\alpha_-(\gamma)}$ or like $d(\cdot, \partial\Omega)|x|^{-\alpha_+(\gamma)}$.

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A Harnack inequality: Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $a \in L^\infty(\Omega)$ and U an open subset of $\mathbb{R}^n$. Consider $u \in C^2(U \cap \overline{\Omega})$ to be a solution of

$$\begin{cases} -\Delta_g u + au = 0 & \text{in } U \cap \Omega \\ u \geq 0 & \text{in } U \cap \Omega \\ u = 0 & \text{on } U \cap \partial\Omega, \end{cases}$$

where g is a smooth metric on U. If $U' \subset\subset U$ is such that $U' \cap \Omega$ is connected, then there exists $C > 0$ depending only on $\Omega, U', \|a\|_\infty$ and g such that

$$\frac{u(x)}{d(x, \partial\Omega)} \leq C \frac{u(y)}{d(y, \partial\Omega)} \text{ for all } x, y \in U' \cap \Omega. \tag{8}$$

- Sub- super-solutions of the linear equations that behave like the two models (one needs to compensate the mean curvature).
- A notion of distributional solutions that distinguish the variational and the non-variational solutions.

What about the case $\gamma > \frac{n^2-1}{4}$?

This involves a notion of mass in the spirit of Schoen-Yau:

Proposition: Fix $\frac{n^2-1}{4} < \gamma < \gamma_H(\Omega)$. Then, up to a positive multiplicative constant, $\exists! G \in C^2(\overline{\Omega} \setminus \{0\})$ such that

$$\left\{ \begin{array}{ll} \Delta G - \frac{\gamma}{|x|^2} G = 0 & \text{in } \Omega \\ G > 0 & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \setminus \{0\} \end{array} \right\}$$

Moreover, there exists $C_1, C_2 \in \mathbb{R}$, $C_1 > 0$, such that

$$G(x) = C_1 \frac{d(x, \partial\Omega)}{|x|^{\alpha_+}} + C_2 \frac{d(x, \partial\Omega)}{|x|^{\alpha_-}} + o\left(\frac{d(x, \partial\Omega)}{|x|^{\alpha_-}}\right)$$

when $x \to 0$. We define the mass as

$$m_\gamma(\Omega) := \frac{C_2}{C_1} \in \mathbb{R}.$$

Is there any domain in $\mathbb{R}^n$ with positive singular mass?

1. The map $\Omega \to m_\gamma(\Omega)$ is a monotone increasing function on the class of domains having zero on their boundary, once ordered by inclusion.
2. $m_\gamma(\mathbb{R}^n_+) = 0$ for any $\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$, and therefore the mass of any one of its subsets having zero on its boundary is non-positive.
   In particular, $m_\gamma(\Omega) < 0$ whenever Ω is convex and $0 \in \partial\Omega$.
3. On the other hand, we have examples of bounded domains Ω in $\mathbb{R}^n$ with $0 \in \partial\Omega$ and with positive mass $m_\gamma(\Omega) > 0$.
4. We have examples of domains with positive/negative mass with any local behavior at 0.

In other words, the sign of the Hardy b-mass is totally independent of the local properties of $\partial\Omega$ around 0.

The remaining case, i.e., $n = 3$ and $s = 0$ and $\gamma \in (0, \frac{n^2}{4})$

In this situation, there may or may not be extremals for $\mu_{\gamma,0}(\mathbb{R}^n_+)$.

1. If they do exist, one can argue as before –using the same test functions– to get:
   - Either $\gamma \leq \frac{n^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative
   - Or $\gamma > \frac{n^2-1}{4}$ and the mass $m_\gamma(\Omega)$ is positive.
2. If no extremal exist for $\mu_{\gamma,0}(\mathbb{R}^n_+)$, then $\mu_{\gamma,0}(\mathbb{R}^n_+) = \mu_{0,0}(\mathbb{R}^n_+)$, the best constant in the Sobolev inequality. We are back to the case of the Yamabe problem with no boundary singularity.

One then resorts to a more standard notion of mass $R_\gamma(\Omega, x_0)$ associated to an interior point $x_0 \in \Omega$ and construct test-functions in the spirit of Schoen: For $\gamma \in (0, \gamma_H(\Omega))$, any solution G of

$$\begin{cases} -\Delta G - \frac{\gamma}{|x|^2} G = 0 & \text{in } \Omega \setminus \{x_0\} \\ G > 0 & \text{in } \Omega \setminus \{x_0\} \\ G = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases}$$

is unique up to multiplication by a constant, and that for any $x_0 \in \Omega$, there exists $R_\gamma(\Omega, x_0) \in \mathbb{R}$ (independent of G) and $c_G > 0$ such that

$$G(x) = c_G \left(\frac{1}{|x - x_0|^{n-2}} + R_\gamma(\Omega, x_0) \right) + o(1) \quad \text{as } x \to x_0.$$

The quantity $R_\gamma(\Omega, x_0)$ is well defined.

Table: The critical cases: $s = 0$

Hardy term	**Dim.**	**Geometric condition**	**Extremal**
$0 < \gamma \leq \frac{n^2-1}{4}$	$n = 3$	Negative mean curvature at 0 & Positive internal mass	Yes
	$n \geq 4$	Negative mean curvature at 0	Yes
$\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$	$n = 3$	Positive boundary-mass & Positive internal mass	Yes
	$n \geq 4$	Positive boundary mass	Yes
$\gamma \leq 0$	$n \geq 3$	–	No