Gradient interfaces with and without disorder

Codina Cotar

University College London

September 09, 2014, Toronto

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

- Microscopic model $\leftrightarrow$ emerging macroscopic structures.
- Macroscopic phases $\rightarrow$ microscopic interfaces

- Approach: Microscopic modelling of the interface itself.

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

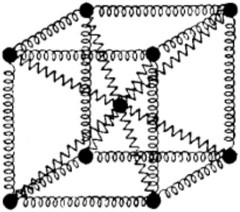
2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

- Crystals are macroscopic objects, with ordered arrangements of atoms or molecules in microscopic scale
- Mechanical model of a crystal: little balls connected by springs, where heat causes the jiggling



- Configuration: snapshot of the atoms' positions at a given time.

- In thermal equilibrium, the jigglings explore samples of a probability measure on the configurations. This is the Gibbs measure:

$$\mathrm{Prob}(\mathrm{Configuration}) \propto \exp(-\beta \text{ Energy of Configuration}),$$

  where $\beta = 1/\mathrm{temperature} > 0$.
- Moving every atom in the same direction the same amount does not change the energy, and hence the probability, of the configuration (shift-invariance).
- If Hook's law holds, the elastic energy between two atoms with displacements x, y is given by $c(x-y)^2$ (the force F needed to extend or compress a spring by some distance $|x-y|$ is proportional to that distance).
- Then the measure on the atoms' configurations is multi-dimensional Gaussian.

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

1D Gaussian random variables

- Recall: A standard 1D Gaussian random variable X has distribution given by the density

$$\mathbb{P}(X \in [x, x+dx]) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx.$$

Gaussian random variables in $\mathbb{R}^n$

- If If $\langle x, y \rangle$ is an inner product in $\mathbb{R}^n$, then

$$(2\pi)^{-n/2} \exp\left(\frac{\langle x, x \rangle}{2}\right)$$

is the density of an associated multidimensional Gaussian.

- This is the same as taking

$$\sum_{j=1}^{n} z_j e_j$$

where $\{e_j\}$ is an orthonormal basis and $\{z_j\}$ are independent 1D Gaussians.

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

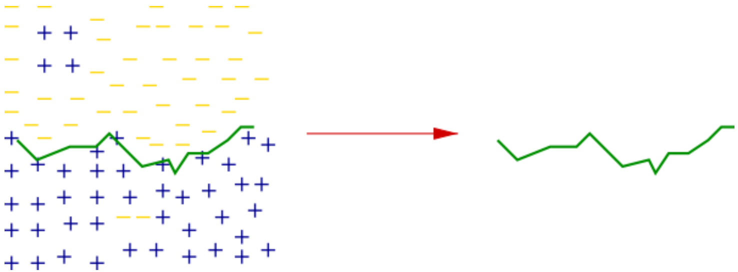
5 Open questions: non-convex potentials

- The interface for the Ising model - simplest description of ferromagnetism
- The spontaneous magnetization on cooling down the substance below a critical temperature, the so-called Curie temperature.
- The Ising model on a domain $\Omega \subset \mathbb{Z}^d$ with free boundary condition, at inverse temperature $\beta = 1/T > 0$ and external field $h \in \mathbb{R}$, is given by the following Gibbs measure on spin configurations $(\sigma_x)_{x\in\Omega} \in \{\pm 1\}^\Omega$

$$\mathbb{P}_{\Omega,h,\beta}(\sigma) := \frac{1}{Z_{\Omega,h,\beta}} \exp\Bigg(\beta \sum_{\substack{x,y\in\Omega \\ |x-y|=1}} \sigma_x\sigma_y + h\sum_{x\in\Omega}\sigma_x\Bigg)\mathbb{P}(\sigma),$$

where $\mathbb{P}$ is the uniform distribution on $\{\pm 1\}^\Omega$.

- Assume $d = 2$ and $\Omega = [0, N] \times [0, N]$.
- Spin configuration $\sigma = \{\sigma_x\}_{x \in \{0,\ldots,N\} \times \{0,\ldots,N\}}$, spins $\sigma_x \in \{-1, 1\}$



- Goal: Modelling and analysis of the interface phase boundary

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

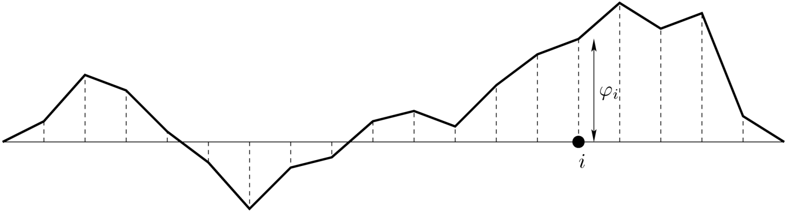
2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

- Interface — transition region that separates different phases



- $\Lambda_n := \{-n, -n+1, \dots, n-1, n\}$, $\partial\Lambda_n = \{-n-1, n+1\}$
- Height Variables (configurations) $\phi_i \in \mathbb{R}, i \in \Lambda_n$
- Boundary condition 0, such that

$$\phi_i = 0, \quad \text{when} \quad i \in \partial\Lambda_n.$$

- The energy $H(\phi) := \sum_{i=-n}^{n+1} V(\phi_i - \phi_{i-1})$, with $V(s) = s^2$ for Hooke's law.

- The finite volume Gibbs measure

$$\nu^0_{\Lambda_n}(\phi_{-n},\dots,\phi_1,\dots,\phi_n) = \frac{1}{Z^0_{\Lambda_n}}\exp(-\beta H(\phi))d\phi_{\Lambda_n} = \frac{1}{Z^0_{\Lambda_n}}\exp(-\beta\sum_{i=-n}^{n+1}(\phi_i-\phi_{i-1})^2)\prod_{i=-n}^{n}d\phi_i,$$

where $\beta = 1/T > 0$, $\phi_{-n-1} = \phi_{n+1} = 0$ and

$$Z^0_{\Lambda_n} := \int_{\mathbb{R}^{2n+1}}\exp(-\beta\sum_{i=-n}^{n+1}(\phi_i-\phi_{i-1})^2)\prod_{i=-n}^{n}d\phi_i,$$

is a multidimensional centered Gaussian measure.

- We can replace the 0-boundary condition in $\nu^0_{\Lambda_n}$ by a ψ-boundary condition in $\nu^{\psi}_{\Lambda_n}$ with $\phi_{-n-1} := \psi_{-n-1}, \phi_{n+1} := \psi_{n+1}$.

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

- Replace the discrete interval $\{-n, -n+1, \ldots, 1, 2, \ldots, n\}$ by a discrete box

$$\Lambda_n := \{-n, -n+1, \ldots, 1, \ldots, n-1, n\}^d,$$

with boundary

$$\partial\Lambda_n := \{i \in \mathbb{Z}^d \setminus \Lambda_n : \exists j \in \Lambda_n \text{ with } |i-j| = 1\}.$$

- The energy $H(\phi) := \sum_{\substack{i,j \in \Lambda_n \cup \partial\Lambda_n \\ |i-j|=1}} V(\phi_i - \phi_j)$, where $V(s) = s^2$ and $\phi_i = 0$ for $i \in \partial\Lambda_n$.
- The corresponding finite volume Gibbs measure on $\mathbb{R}^{\Lambda_n}$ is given by

$$\nu^0_{\Lambda_n}(\phi) := \frac{1}{Z_{\Lambda_n}} \exp(-\beta H(\phi)) \prod_{i \in \Lambda_n} d\phi_i.$$

It is a Gaussian measure, called the Gaussian Free Field (GFF).

For GFF

- If $x, y \in \Lambda_n$
$$\operatorname{cov}_{\nu^0_{\Lambda_n}}(\phi_x, \phi_y) = G_{\Lambda_n}(x, y),$$
where $G_{\Lambda_n}(x, y)$ is the Green's function, that is, the expected number of visits to y of a simple random walk started from x killed when it exits Λ_n.
- GFF appears in many physical systems; two-dimensional GFF has close connections to Schramm-Loewner Evolution (SLE).
- Random, fractal curve in $\Omega \subseteq \mathbb{C}$ simply connected.
- Introduced by Oded Schramm as a candidate for the scaling limit of loop erased random walk (and the interfaces in critical percolation).
- Contour lines of the GFF converge to SLE (Schramm-Sheffield 2009).

General potential V, general boundary condition ψ, general Λ

- $V : \mathbb{R} \to \mathbb{R}, V \in C^2(\mathbb{R})$ with $V(s) \geq As^2 + B, A > 0, B \in \mathbb{R}$ for large s.
- The finite volume Gibbs measure on $\mathbb{R}^\Lambda$

$$\nu_\Lambda^\psi(\phi) := \frac{1}{Z_\Lambda^\psi} \exp\Big(-\beta \sum_{\substack{i,j \in \Lambda \cup \partial\Lambda \\ |i-j|=1}} V(\phi_i - \phi_j)\Big) \prod_{i \in \Lambda} d\phi_i,$$

  where $\phi_i = \psi_i$ for $i \in \partial\Lambda$.
- tilt $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ and tilted boundary condition $\psi_i^u = i \cdot u, i \in \partial\Lambda$.
- Finite volume surface tension (free energy) $\sigma_\Lambda(u)$: macroscopic energy of a surface with tilt $u \in \mathbb{R}^d$.

$$\sigma_\Lambda(u) := \frac{1}{|\Lambda|} \log Z_\Lambda^{\psi^u}.$$

- Gradients $\nabla\phi$: $\nabla\phi_b = \phi_i - \phi_j$ for $b = (i,j), |i-j| = 1$

Questions (for general potentials V):

- Existence and (strict) convexity of infinite volume (i.e., infinite dimensional) surface tension

$$\sigma(u) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \sigma_\Lambda(u).$$

- Existence of shift-invariant infinite dimensional Gibbs measure

$$\nu := \lim_{\Lambda \uparrow \mathbb{Z}^d} \nu_\Lambda^\psi$$

- Uniqueness of shift-invariant Gibbs measure under additional assumptions on the measure.
- Quantitative results for ν: decay of covariances with respect to ϕ, central limit theorem (CLT) results, log-Sobolev inequalities, large deviations (LDP) results.

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

Known results for potentials V with

$$0 < C_1 \leq V'' \leq C_2 :$$

- Existence and strict convexity of the surface tension σ for $d \geq 1$ and $\sigma \in C^1(\mathbb{R}^d)$.
- Gibbs measures ν do not exist for $d = 1, 2$.
- We can consider the distribution of the $\nabla\phi$-field under the Gibbs measure ν. We call this measure the $\nabla\phi$-Gibbs measure μ.
- $\nabla\phi$-Gibbs measures μ exist for $d \geq 1$.
- (Funaki-Spohn (CMP-2007)) For every $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ there exists a **unique shift-invariant ergodic** $\nabla\phi$- Gibbs measure μ with $E_\mu[\phi_{e_k} - \phi_0] = u_k$, for all $k = 1, \ldots, d$.
- CLT results, LDP results

Bolthausen, Brydges, Deuschel, Funaki, Giacomin, Ioffe, Naddaf, Olla, Peres, Sheffield, Spencer, Spohn, Velenik, Yau, Zeitouni

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

For

$$0 < C_1 \leq V'' \leq C_2 :$$

- Brascamp-Lieb Inequality (Brascamp-Lieb JFA 1976/Caffarelli-CMP 2000): for all $x \in \Lambda$ and for all $i \in \Lambda$

$$\text{var}_{\nu_\Lambda^\psi}(\phi_i) \leq \text{var}_{\tilde{\nu}_\Lambda^\psi}(\phi_i),$$

  $\tilde{\nu}_\Lambda^\psi$ is the Gaussian Free Field with potential $\tilde{V}(s) = C_1 s^2$.
- Random Walk Representation (Deuschel-Giacomin-Ioffe 2000): Representation of Covariance Matrix in terms of the Green function of a particular random walk.
  - GFF: If $x, y \in \Lambda$

$$\text{cov}_{\nu_\Lambda^0}(\phi_x, \phi_y) = G_\Lambda(x, y).$$

  - General $0 < C_1 \leq V'' \leq C_2$:
    $0 \leq \text{cov}_{\nu_\Lambda^\psi}(\phi_x, \phi_y) \leq \frac{C}{]|x-y|[^{d-2}}$, $|\text{cov}_{\mu_\Lambda^\rho}(\nabla_i \phi_x, \nabla_j \phi_y)| \leq \frac{C}{]|x-y|[^{d-2+\delta}}$

- The dynamic: SDE satisfied by $(\phi_x)_{x\in\mathbb{Z}^d}$

$$d\phi_x(t) = -\frac{\partial H}{\partial \phi_x}(\phi(t))dt + \sqrt{2}dW_x(t), \quad x \in \mathbb{Z}^d,$$

where $W_t := \{W_x(t), x \in \mathbb{Z}^d\}$ is a family of independent 1-dim Brownian Motions.

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

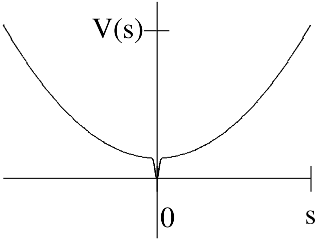
Why look at the case with non-convex potential V?

- Probabilistic motivation: Universality class
- Physics motivation: For lattice spring models a realistic potential has to be non-convex to account for the phenomena of fracturing of a crystal under stress.
- The Cauchy-Born rule: When a crystal is subjected to a small linear displacement of its boundary, the atoms will follow this displacement.
- Friesecke-Theil: for the 2-dimensional mass-spring model, Cauchy-Born holds for a certain class of non-convex potentials. Generalization to d-dimensional mass-spring model by Conti, Dolzmann, Kirchheim and Müller.

Results for non-convex potentials

- For the potential

$$e^{-V(s)} = pe^{-k_1\frac{s^2}{2}} + (1-p)e^{-k_2\frac{s^2}{2}}, \ \beta = 1, k_1 << k_2, \ p = \left(\frac{k_1}{k_2}\right)^{1/4}$$



- Biskup-Kotecký (PTRF-2007): Existence of several $\nabla\phi$-Gibbs measures with expected tilt $E_\mu[\phi_{e_k} - \phi_0] = 0$, but with different variances.

- Cotar-Deuschel-Müller (CMP-2009)/ Cotar-Deuschel (AIHP-2012):
  Let
  $$V = V_0 + g,\ C_1 \leq V_0'' \leq C_2,\ g'' < 0.$$
  If
  $$C_0 \leq g'' < 0 \ \text{ and } \quad \sqrt{\beta}||g''||_{L^1(\mathbb{R})} \ \text{small}(C_1, C_2)$$
  uniqueness for shift-invariant $\nabla\phi$-Gibbs measures μ such that $E_\mu\left[\phi_{e_k} - \phi_0\right] = u_k$ for $k = 1, 2, \ldots, d$. Our results includes the Biskup-Kotecký model, but for different range of choices of p, k_1 and k_2.
- Adams-Kotecký-Müller (preprint): Strict convexity of the surface tension for very small tilt u and very large β.

Outline

1 Physics motivation
- Example 1: Elasticity
- Recap-Gaussian Measure
- Example 2: Effective interface models

2 The model
- Dimension $d = 1$
- Generalization to dimension $d \geq 2$

3 Questions

4 Known results
- Results: Strictly Convex Potentials
- Techniques: Strictly Convex Potentials
- Results: Non-convex potentials
- Interfaces with disorder

5 Open questions: non-convex potentials

- Adding disorder (for example, making potentials random variables) tends to destroy non-uniqueness.
- Consider for simplicity the disordered model

$$e^{-V_b(\eta_b)} := pe^{-k_1(\eta_b)^2+\omega_b}+(1-p)e^{-k_2(\eta_b)^2-\omega_b}, \ (w_b)_b \text{ i.i.d. Bernoulli.}$$

  Adaptation of the Aizenman-Wehr (CMP-1990) argument: gives uniqueness of gradient Gibbs in $d = 2$
- Conjecture
  - uniqueness for low enough $d \leq d_c$;
  - uniqueness/non-uniqueness phase transition for high enough $d > d_c \geq 2$.
- Techniques: Poincarre inequalities (Gloria/Otto), log-Sobolev inequalities (Milman 2012).

- Log-Sobolev inequality for moderate/low temperature.
- Relaxation of the Brascamp-Lieb inequality.
- Example of potential where the surface tension is non-strictly-convex.
- Conjecture: Surface tension (plus maybe some additional assumption) $\Rightarrow$ uniqueness of the shift-invariant Gibbs measure.
- Conjecture: Surface tension is in $C^2(\mathbb{R}^d)$ (both for strictly convex and for non-convex potentials).

THANK YOU!