Integrability of pentagram maps and Lax representations

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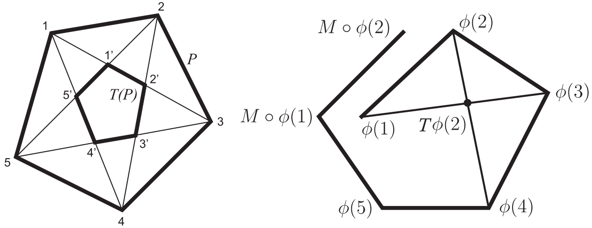
2D case (S'92; OST'10)

2D pentagram map:



Closed and twisted pentagons.

The 2D pentagram map is defined as $T\phi(j) := (\phi(j-1), \phi(j+1)) \cap (\phi(j), \phi(j+2))$. Choosing appropriate lifts of the points $\phi(j)$ to the vectors V_j in $\mathbb{C}^3$, we can associate a difference equation

$$V_{j+3} = a_{j,2}V_{j+2} + a_{j,1}V_{j+1} + V_j.$$

Transformations $T^*(a_{j,1})$ and $T^*(a_{j,2})$ are rational functions in $a_{*,1},\ a_{*,2}$.

Continuous limit in the 2D case

In the continuous case, we have a 3rd order linear ordinary differential equation instead of the difference equation $V_{j+3} = a_j V_{j+2} + b_j V_{j+1} + V_j$. The normalization condition $\det(V_j, V_{j+1}, V_{j+2}) = 1$ corresponds to the choice of solutions having the unit Wronskian. More precisely, we have:

Theorem 1

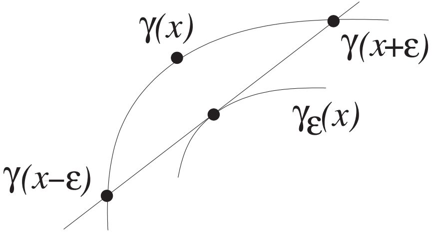
There is a one-to one correspondence between equivalence classes of non-degenerate curves in $\mathbb{CP}^2$ ($\mathbb{RP}^2$) and operators

$$L = \partial_x^3 + a_1(x)\partial_x + a_0(x),$$

where $a_1(x), a_0(x)$ are smooth functions.

Continuous limit in the 2D case

The envelope of the chords $(\gamma(x-\varepsilon), \gamma(x+\varepsilon))$ for different x leads to a new curve $\gamma_\varepsilon(x)$:



Theorem 2

The corresponding differential operator equals $L_\varepsilon = L + \varepsilon^2[Q_2, L] + O(\varepsilon^3)$, *where* $Q_2 = (L^{2/3})_+ = \partial^2 + (2/3)a_1(x)$. *The equation* $\dot{L} = [Q_2, L]$ *is equivalent to the Boussinesq equation.*

Definitions

A twisted n-gon is a map $\phi : \mathbb{Z} \to \mathbb{P}^d$, such that $\phi(k+n) = M \circ \phi(k)$ for any k, and $M \in PSL_{d+1}$. M is called the monodromy. None of the $d+1$ consecutive vertices lie on one hyperplane $\mathbb{P}^{d-1}$. Two twisted n-gons are equivalent if there is a transformation $g \in PSL_{d+1}$, such that $g \circ \phi_1 = \phi_2$.
The dimension of the space of polygons is

$$\dim \mathcal{P}_n = nd + \dim SL_{d+1} - \dim SL_{d+1} = nd.$$

One can show that there exists a unique lift of the vertices $v_k = \phi(k) \in \mathbb{P}^d$ to the vectors $V_k \in \mathbb{C}^{d+1}$ satisfying $\det(V_j, V_{j+1}, ..., V_{j+d}) = 1$ and $V_{j+n} = MV_j,\ j \in \mathbb{Z}$, where $M \in SL_{d+1}$ (provided that $gcd(n, d+1) = 1$).
When $gcd(n, d+1) = 1$, difference equations with n-periodic coefficients in j:

$$V_{j+d+1} = a_{j,d} V_{j+d} + a_{j,d-1} V_{j+d-1} + ... + a_{j,1} V_{j+1} + (-1)^d V_j, \quad j \in \mathbb{Z},$$

allow one to introduce coordinates $\{a_{j,k},\ 0 \le j \le n-1,\ 1 \le k \le d\}$ on the space $\mathcal{P}_n$.

Definitions

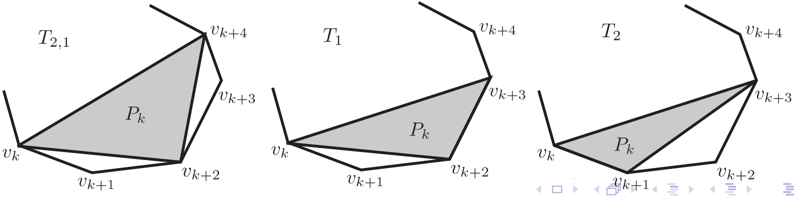
For a $(d-1)$-tuple of jumps (positive integers) $I = (i_1, i_2, \ldots, i_{d-1})$ an I-diagonal hyperplane is $P_k := (v_k, v_{k+i_1}, v_{k+i_2}, \ldots, v_{k+i_{d-1}})$.
Generalized pentagram map in $\mathbb{P}^d$ is
$Tv_k := P_k \cap P_{k+1} \cap \ldots \cap P_{k+d-1}$. Clearly, this definition is projectively invariant.

We discovered several integrable cases:

(a) "Short-diagonal": $I = (2, 2, \ldots, 2)$ (KS for $d = 3$, Mari-Beffa for higher d)

(b) "Dented": $I_m = I = (1, \ldots, 1, 2, 1, \ldots, 1)$ (the only 2 is at the m-th place; $1 \le m \le d-1$ is an integer parameter).

(c) "Deep-dented": $I_m^p = I = (1, \ldots, 1, p, 1, \ldots, 1)$ (the number p is at the m-th place; it has 2 integer parameters m and p).



Lax representation

A Lax representation is a compatibility condition for an over-determined system of linear equations.

Example.

$$\begin{cases} L\psi = k\psi \\ P\psi = \partial_t \psi \end{cases} \Leftrightarrow \partial_t L = [P, L].$$

As a consequence, $d(\mathrm{tr}\, L^j)/dt = 0$ for any j. If L is an $n \times n$ matrix, we have n conserved quantities.

If L, P depend on an auxiliary parameter λ, we may have more.

A discrete zero-curvature equation is a compatibility condition for

$$\begin{cases} L_{i,t}(\lambda)\psi_{i,t}(\lambda) = \psi_{i+1,t}(\lambda) \\ P_{i,t}(\lambda)\psi_{i,t}(\lambda) = \psi_{i,t+1}(\lambda) \end{cases} \Leftrightarrow L_{i,t+1}(\lambda) = P_{i+1,t}(\lambda)L_{i,t}(\lambda)P^{-1}_{i,t}(\lambda)$$

$$\begin{array}{ccccccccc}
\psi_{i,t+1} & \xrightarrow{L_{i,t+1}} & \psi_{i+1,t+1} & \longrightarrow & \dots & \longrightarrow & \psi_{i+n-1,t+1} & \xrightarrow{L_{i+n-1,t+1}} & \psi_{i+n,t+1} \\
P_{i,t}\Big\uparrow & & P_{i+1,t}\Big\uparrow & & & & P_{i+n-1,t}\Big\uparrow & & P_{i+n,t}\Big\uparrow \\
\psi_{i,t} & \xrightarrow{L_{i,t}} & \psi_{i+1,t} & \longrightarrow & \dots & \longrightarrow & \psi_{i+n-1,t} & \xrightarrow{L_{i+n-1,t}} & \psi_{i+n,t}
\end{array}$$

Lax representation

Theorem 3

In 3D case, i.e., when $d = 3$, we have:

(a) *"Short-diagonal" case:* $L_{i,t}(\lambda) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ \lambda & 0 & 0 & a_{i,1} \\ 0 & 1 & 0 & a_{i,2} \\ 0 & 0 & \lambda & a_{i,3} \end{pmatrix}^{-1}$

(b) *"Dented" case:* $L_{i,t}(\lambda) = \left(\begin{array}{c|c} 0 \quad 0 \quad 0 & -1 \\ \hline D(\lambda) & \begin{matrix} a_{i,1} \\ a_{i,2} \\ a_{i,3} \end{matrix} \end{array}\right)^{-1}$,

where $D(\lambda) = diag\,(1, \lambda, 1)$ or $D(\lambda) = diag\,(1, 1, \lambda)$ (λ is situated at the $(m+1)$-th place)

(c) *The "deep-dented" case is more complicated, the Lax function has the size $(p+2) \times (p+2)$.*

In each case there exists a corresponding function $P_{i,t}$.

AG integrability

Definition 4

Monodromy operators $T_{0,t}$, $T_{1,t}$, ..., $T_{n-1,t}$ are defined as the following ordered products of the Lax functions:

$$
\begin{aligned}
T_{0,t} &= L_{n-1,t}L_{n-2,t}...L_{0,t},\\
T_{1,t} &= L_{0,t}L_{n-1,t}L_{n-2,t}...L_{1,t},\\
T_{2,t} &= L_{1,t}L_{0,t}L_{n-1,t}L_{n-2,t}...L_{2,t},\\
...\\
T_{n-1,t} &= L_{n-2,t}L_{n-3,t}...L_{0,t}L_{n-1,t}.
\end{aligned}
$$

A Floquet-Bloch solution $\psi_{i,t}$ of a difference equation $\psi_{i+1,t} = L_{i,t}\psi_{i,t}$ is an eigenvector of the monodromy operator: $T_{i,t}\psi_{i,t} = w\psi_{i,t}$.

A normalization of the vector $\psi_{0,0}$ determines $\psi_{i,t}$ uniquely: $\sum_{j=1}^{4} \psi_{0,0,j} \equiv 1$.

The spectral curve is defined by $R(w,\lambda) = \det\left(T_{i,t}(\lambda) - w \cdot Id\right)$.

AG integrability

Theorem 5

$R(w,\lambda)$ does not depend on i,t.
Generically, in the cases (a) and (b), $R(w,\lambda)=0$ defines a Riemann surface Γ of genus $g=3q$ for odd n and $g=3q-3$ for even n, where $q=\lfloor n/2 \rfloor$.
A Floquet-Bloch solution $\psi_{i,t}$ is a meromorphic vector function on Γ.
Generically, its pole divisor $D_{i,t}$ has degree $g+3$.

Remark. The coefficients of $R(w,\lambda)$ are integrals of motion.

Definition 6

The spectral data consists of the generic spectral curve Γ with marked points and a point $[D]$ in its Jacobian $J(\Gamma)$.
The map $S:\mathcal{P}_n \to (\Gamma,[D_{0,0}],\text{marked points})$ is called the direct spectral transform.
The map $S_{inv}:(\Gamma,[D],\text{marked points}) \to \mathcal{P}_n$ is called the inverse spectral transform.

AG integrability

Theorem 7

Both maps S and S_{inv} are defined on Zariski open subsets. $S \circ S_{inv} = Id$ and $S_{inv} \circ S = Id$ whenever the composition is defined.

Remark. Now the independence of the first integrals follows from the dimension counting.

Main example in this talk: short-diagonal case.

$$R(w,\lambda) = w^4 - w^3 \left(\sum_{j=0}^{q} G_j \lambda^{j-n} \right) + w^2 \left(\sum_{j=0}^{q} J_j \lambda^{j-q-n} \right) - \\ - w \left(\sum_{j=0}^{q} I_j \lambda^{j-2n} \right) + \lambda^{-2n}.$$

Properties of the spectral curve

Theorem 8 (short-diagonal case)

Generically, the genus of the spectral curve Γ is $g = 3q$ for odd n and $g = 3q - 3$ for even n, where $q = \lfloor n/2 \rfloor$. It has 5 marked points for odd n (denoted by O_1, O_2, O_3, W_1, W_2) and 8 marked points for even n $(O_1, O_2, O_3, O_4, W_1, W_2, W_3, W_4)$. The corresponding Puiseux series for even n at $\lambda = 0$ are

$$O_1: \quad w_1 = \frac{1}{I_0} - \frac{I_1}{I_0^2}\lambda + \mathcal{O}(\lambda^2),$$

$$O_{2,3}: \quad w_{2,3} = \frac{w_*}{\lambda^q} + \mathcal{O}\left(\frac{1}{\lambda^{q-1}}\right), \quad \text{where} \quad G_0 w_*^2 - J_0 w_* + I_0 = 0,$$

$$O_4: \quad w_4 = \frac{G_0}{\lambda^n} + \frac{G_1}{\lambda^{n-1}} + \frac{G_2}{\lambda^{n-2}} + \mathcal{O}(\lambda^{3-n}),$$

And at $\lambda = \infty$ they are

$$W_*: \; w_{1,2,3,4} = \frac{w_\infty}{\lambda^q} + \mathcal{O}\left(\frac{1}{\lambda^{q+1}}\right), \; w_\infty^4 - G_q w_\infty^3 + J_q w_\infty^2 - I_q w_\infty + 1 = 0.$$

Properties of the spectral curve

The Puiseux series for odd n at $\lambda = 0$ are

$$O_1: \quad k_1 = \frac{1}{I_0} - \frac{I_1}{I_0^2}\lambda + \mathcal{O}(\lambda^2),$$

$$O_2: \quad k_{2,3} = \pm\frac{\sqrt{-I_0/G_0}}{\lambda^{n/2}} + \frac{J_0}{2G_0\lambda^{(n-1)/2}} + \mathcal{O}\left(\frac{1}{\lambda^{(n-2)/2}}\right),$$

$$O_3: \quad k_4 = \frac{G_0}{\lambda^n} + \frac{G_1}{\lambda^{n-1}} + \frac{G_2}{\lambda^{n-2}} + \mathcal{O}(\lambda^{3-n}),$$

And at $\lambda = \infty$ they are

$$W_{1,2}: \quad k_{1,2,3,4} = \frac{k_\infty}{\lambda^{n/2}} + \mathcal{O}\left(\frac{1}{\lambda^{(n+1)/2}}\right), \text{ where } k_\infty^4 + J_q k_\infty^2 + 1 = 0.$$

AG integrability

Theorem 9 (short-diagonal case)

- *when n is odd,*

$$[D_{0,t}] = [D_{0,0} - tO_{13} + tW_{12}],$$

- *when n is even,*

$$[D_{0,t}] = \left[D_{0,0} - tO_{14} + \lfloor\frac{t}{2}\rfloor W_{12} + \lfloor\frac{t+1}{2}\rfloor W_{34}\right].$$

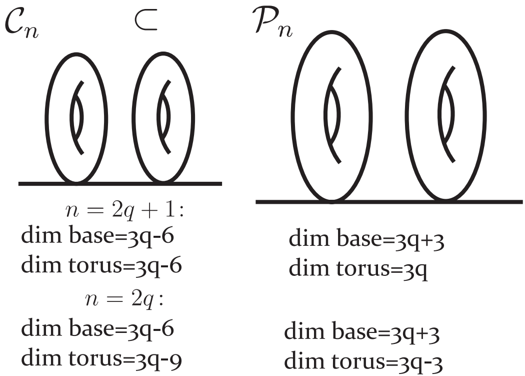
(We denote $O_{pq} := O_p + O_q$ and $W_{pq} := W_p + W_q$).

Integrability for closed polygons

Closed polygons in $\mathbb{CP}^3$ correspond to the monodromies $M = \pm \mathrm{Id}$ in $SL(4, \mathbb{C})$. They form a subspace $\mathcal{C}_n$ of codimension $15 = \dim SL(4, \mathbb{C})$ in the space of all twisted polygons $\mathcal{P}_n$.

Theorems 7 and 9 hold verbatim for closed manifolds.

The genus of Γ drops by 6 for closed polygons, because $M \equiv T_{0,0}|_{\lambda=1}$.



The symplectic form

Definition 10

Krichever-Phong's universal formula defines a pre-symplectic form on the space $\mathcal{P}_n$. It is given by the expression:

$$\omega = -\frac{1}{2} \sum_{\lambda=0,\infty} \operatorname{res} \operatorname{Tr} \left(\Psi_{0,0}^{-1} T_{0,0}^{-1} \delta T_{0,0} \wedge \delta \Psi_{0,0} \right) \frac{d\lambda}{\lambda},$$

where the matrix $\Psi_{0,0}(\lambda)$ consists of the vectors $\psi_{0,0}$ taken on different sheets of Γ.

The leaves of the 2-form ω are defined as submanifolds of $\mathcal{P}_n$, where the expression $\delta \ln w d\lambda/\lambda$ is holomorphic. The latter expression is considered as a one-form on the spectral curve Γ.

The symplectic form

Theorem 11 (short-diagonal case)

For even n the leaves are singled out by 6 conditions:

$$\delta I_0 = \delta I_q = \delta G_0 = \delta G_q = \delta J_0 = \delta J_q = 0;$$

For odd n the leaves are singled out by 3 conditions:

$$\delta G_0 = \delta I_0 = \delta J_q = 0.$$

When restricted to the leaves, ω becomes a symplectic form of rank $2g$, invariant w.r.t the pentagram map.

Remark. This theorem implies Arnold-Liouvile integrability (in a generalized sense).

The symplectic form

Theorem 12 (Action-angle variables)

Let the divisor of poles of $\psi_{0,0}$ *on* Γ *be* $D_{0,0} = \sum_{s=1}^{g+3} \gamma_s$. *When restricted to the leaves,*

$$\omega = \sum_{i=1}^{g+3} \delta \ln w(\gamma_i) \wedge \delta \ln \lambda(\gamma_i) = \sum_{i=1}^{g} \delta \mathbf{I}_i \wedge \delta \varphi_i,$$

$$\text{where } \mathbf{I}_i = \oint_{a_i} \ln w d\lambda/\lambda, \quad \varphi_i = \sum_{s=1}^{g+3} \int^{\gamma_s} d\omega_i,$$

and one-forms $d\omega_i$, $1 \leq i \leq g$, *form a basis of* $H^0(\Gamma, \Omega^1)$.

Dynamics of the pentagram maps

Theorem 13

The above integrable pentagram maps on twisted n-gons in $\mathbb{CP}^d$ cannot be included into a Hamiltonian flow as its time-one map, at least for some values of n, m, and d.

This suggests the following

Definition 14

Suppose that (M, ω) is a $2n$-dimensional symplectic manifold and $I_1, ..., I_n$ are n independent functions in involution. Let $M_{\mathbf{c}}$ be a (possibly disconnected) level set of these functions: $M_{\mathbf{c}} = \{x \in M \mid I_j(x) = c_j,\ 1 \leq j \leq n\}$. A map $T : M \to M$ is called generalized integrable if

- it is symplectic, i.e., $T^*\omega = \omega$;
- it preserves the integrals of motion: $T^* I_j \equiv I_j,\ 1 \leq j \leq n$;
- there exists a positive integer $q \geq 1$ such that the map T^q leaves all connected components of level sets $M_{\mathbf{c}}$ invariant for all $\mathbf{c} = (c_1, ..., c_n)$.

References

- B. Khesin, F. Soloviev, *Non-integrability vs. integrability in pentagram maps,* to appear in Journal of Geom. and Physics; arXiv:1404.6221.
- B. Khesin, F. Soloviev, *Integrability of higher pentagram maps,* Math. Ann., vol. 357 (2013), no.3, 1005–1047; arXiv:1204.0756.
- B. Khesin, F. Soloviev, *The geometry of dented pentagram maps,* to appear in JEMS (2013), 32pp.; arXiv:1308.5363.
- F. Soloviev, *Integrability of the pentagram map,* Duke Math. Journal, vol. 162 (2013), no.15, 2815–2853; arXiv:1106.3950.
- V. Ovsienko, R. Schwartz, S. Tabachnikov, *The pentagram map: a discrete integrable system,* Comm. Math. Phys., vol. 299 (2010), 409–446; arXiv:0810.5605
- R. Schwartz, *The pentagram map,* Experiment. Math., vol. 1 (1992), 71–81.