variational methods for effective dynamics, part II

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Our main concern

If $F_\varepsilon \xrightarrow{\Gamma} F_0$, do the evolution equations

$$\left.\begin{array}{r} \dot{x}_\varepsilon(t) \\ \ddot{x}_\varepsilon(t) \\ \mathbb{J}\dot{x}_\varepsilon(t) \end{array}\right\} = -\nabla F_\varepsilon(x(t))$$

converge to some limiting problem (eg, the $\varepsilon = 0$ evolution problems)?

- for gradient flows, $\exists$ more tools and abstract general theory.
- for Hamiltonian systems, no general theory, but calculus of variations can help:
  - rephrase as dynamic stability problem
  - use variational estimates
  - Strategy: find functionals $\zeta(v; t)$ such that

$$\zeta(v,t) \approx 0 \approx \min \zeta \qquad \text{iff } v(t) \text{ behaves as hoped, and}$$

$$\boxed{\frac{d}{dt}\zeta(v,t) \lesssim \zeta(v,t).}$$

Recall also

- Γ-convergence : general theory, with many examples
- Γ-convergence and gradient flows: general theory, few examples
- Γ-convergence and Hamiltonian systems: no general theory, few examples.

Yesterday we saw an example in which simple variational stability arguments suffice to characterize effective dynamics.

Today: an example in which this is *not* the case.... but more refined variational estimates are useful.

(eg, *quantitative* improvements of Γ-convergence compactness results.)

Today we will focus on

$$E_\varepsilon(v) := \frac{1}{|\log\varepsilon|}\int_{\mathbb{R}^2} \eta^2 \left(\frac{|\nabla v|^2}{2} + \frac{\eta^p}{4\varepsilon^2}(|v|^2-1)^2 \right)$$

for $v \in H^1(\Omega;\mathbb{C})$, where $\Omega \subset \mathbb{R}^2$ and $p \geq 0$; together with

$$i|\log\varepsilon|\partial_t v - \frac{1}{\eta^2}\nabla\cdot(\eta^2\nabla v) + \frac{\eta^p}{\varepsilon^2}(|v|^2-1)v = 0.$$

The main cases of interest are $p = 0, 1$ (although in fact p is basically irrelevant).

The energy is conserved by solutions of the PDE..

We assume that η is fixed, C^2, positive. Still okay if

- $\eta = \eta_\varepsilon > 0$ converges uniformly to a limit,
- limit need only be nonnegative

Motivations

1. The PDE (with $p = 1$) may be obtained by transforming the equation

$$\boxed{i\partial_t u - \Delta u + \frac{1}{\varepsilon^2}\left(V(x) + |u|^2\right) u = 0 \qquad \text{in } \mathbb{R}^2}$$

with $\eta = \eta_\varepsilon$ minimizing

$$\zeta \mapsto \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\zeta|^2 + \frac{1}{\varepsilon^2}\left(V(x)\frac{|u|^2}{2} + \frac{|u|^4}{4}\right) dx$$

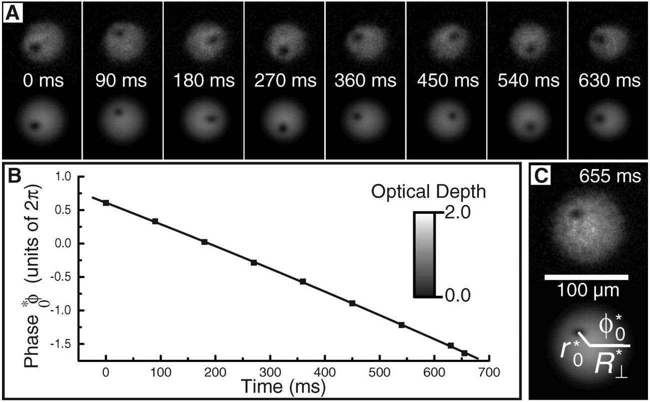
subject to L^2 constraint. Indeed, define v by $u(x,t) = \eta(x)e^{-i\lambda_\varepsilon t}v(x, t|\log\varepsilon|)$.

Describes point vortices in pancake-shaped Bose-Einstein condensates

2. The PDE (with $p = 0$) may be obtained by symmetry reduction from

$$\boxed{i\partial_t u - \Delta u + \frac{1}{\varepsilon^2}\left(|u|^2 - 1\right) u = 0 \qquad \text{in } \mathbb{R}^3}$$

(Write in cylindrical coordinates (r, θ, z), seek solutions independent of θ.) Then $\Omega = (0,\infty)\times\mathbb{R}$ and $\eta^2 = r$. Describes vortex rings in a 3d ideal homgeneous quantum fluid



experimental data showing vortex motion in a Bose-Einstein condensate – vortices precess at constant angular velocity Frelich, Bianchi, Kaufman, Langin and Hall, *Science* 2010

Notation: Given $v \in H^1(\Omega; \mathbb{C})$ we will write

$$j(v) := -\frac{i}{2}(\bar{v}\nabla v - v\bar{\nabla} v) \ := \ \text{momentum density}$$

$$\omega(v) = \frac{1}{2}\nabla \times j(v) \ := \ \text{vorticity}$$

Fact: If $v = \rho e^{i\phi}$ then

$$\boxed{j(v) = \rho^2 \nabla\phi}$$

Fact: If $v = v_1 + iv_2$ then

$$\boxed{\omega(v) = \det(\partial_i v_j) = \text{Jac}(v)}$$

Theorem (J., Sandier, J.-Soner, Alberti-Baldo-Orlandi .... '98-'03)

0. compactness: *Assume that* $(v_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ *and that*

$$E_\varepsilon(v) = \frac{1}{|\log\varepsilon|}\int_{\mathbb{R}^2}\eta^2\left(\frac{|\nabla v|^2}{2} + \frac{\eta^p}{4\varepsilon^2}(|v|^2-1)^2\right) \le C \qquad \text{for all } \varepsilon \in (0,1].$$

Then there exists points $a_i \in \Omega$ *and integers* d_i *such that* $\pi\sum|d_i|\eta^2(a_i) < \infty$ *and after possibly passing to a subsequence*

$$\omega(v_\varepsilon) \to \pi\sum d_i\delta_{a_i} \qquad \text{in } W^{-1,1}, \tag{1}$$

1. *Assume that* $(v_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ *satisfies* (1). *Then*

$$\liminf_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon) \ge \pi\sum|d_i|\eta^2(a_i)$$

2. *For any measure as on the right-hand side of* (1), *there exists a sequence* (v_ε) *such that* (1) *holds and*

$$\limsup_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon) \le \pi\sum|d_i|\eta^2(a_i)$$

About the theorem:

1. Why $\omega(v) \approx \pi \sum d_i \delta_{a_i}$**?**

Main point: Let $S := \{|v| \le 1/2\}$, and assume that

$$S \subset \cup B_i, \qquad B_i := B(x_i, r_i) \text{ with } \deg(v; \partial B_i) =: d_i.$$

Then

$$\boxed{\|\omega(v) - \pi \sum d_i \delta_{x_i}\|_{W^{-1,1}} \le C(\sum r_i) E_\varepsilon(v) |\log \varepsilon|.}$$

2. Why $E_\varepsilon(v) \gtrsim \pi \sum |d_i| \eta^2(a_i)$**?**

Main points:

- model lower bound on balls (*e.g.* equivariant, ie $v = f(r)e^{i\theta}$) is

$$\boxed{\int_{B(s)} \frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}(|v|^2 - 1)^2 \ \ge \pi \log(\frac{r}{\varepsilon}) - O(1)}$$

- there is an algorithm for covering S with balls satisfying comparable lower bound, with tunable $\sum r_i$.

These tools yield more: quantitative estimates for fixed $\varepsilon \ll 1$.

Theorem (J.-Smets '13)

Let v_ε be a sufficiently smooth solution of

$$i|\log\varepsilon|\partial_t v_\varepsilon - \frac{1}{\eta^2}\nabla\cdot(\eta^2\nabla v_\varepsilon) + \frac{\eta^p}{\varepsilon^2}(|v_\varepsilon|^2-1)v_\varepsilon = 0.$$

with initial data v_ε^0 such that

$$\omega(v_\varepsilon^0) \to \pi\sum d_i\delta_{a_i^0}, \qquad E_\varepsilon(v_\varepsilon)\to\pi\sum\eta^2(a_i^0)$$

with $|d_i|=1$ for all i. i.e. *a recovery sequence for the measure $\pi\sum d_i\delta_{a_i^0}$. Then*

$$\omega(v_\varepsilon(t)) \to \pi\sum d_i\delta_{a_i(t)},$$

where each $a_i(t)$ solves

$$\dot a_i(t) = d_i\nabla^\perp\log\eta^2(a_i), \qquad a_i(0)=a_i^0.$$

This result is valid as long as no two points $a_i(\cdot)$ collide.

The case $\eta =$ constant is easier and has been understood since the late 90s, see Colliander-J, Lin-Xin, Spirn, Gustafson-Sigal, J-Spirn, ...

Our proof takes some ingredients from from some of these, particularly J-Spirn .

The following discussion (except at the last slide) emphasizes new points.

Heuristics

We will study the PDE for $\varepsilon \ll 1$ fixed, and we write v instead of v_ε.

- We need to understand evolution of $\omega(v)$.
- Evolution of $\omega(v)$ governed by identity (in integral form)

$$\frac{d}{dt}\int_\Omega \varphi\omega(v) = \frac{1}{|\log\varepsilon|}\int_\Omega \varepsilon_{lj}\varphi_{x_l}\frac{\eta^2_{x_k}}{\eta^2}\left[v_{x_j}\cdot v_{x_k} + \delta_{jk}\frac{\eta^2}{\varepsilon^2}(|v|^2-1)^2\right] + \varepsilon_{lj}\varphi_{x_k x_l} v_{x_j}\cdot v_{x_k}$$

- Green term is lower-order.
- If φ is linear near vortices, then blue term is lower-order.
- We need $\boxed{\dfrac{v_{x_i}\cdot v_{x_j}}{|\log\varepsilon|} \approx \pi\delta_{ij}\sum \delta_{\xi_i(t)}}$, where $\xi_i(t) \approx$ vortex locations.

More heuristics Let us suppose that quantitative versions of Γ-limit theorem hold for fixed $\varepsilon > 0$. This is in fact the case.

- Quantitative compactness should imply: there exist points $\xi_i(t) =$"actual vortex locations" such that

$$\|\omega(v(t)) - \pi \sum d_i \delta_{\xi_i(t)}\|_{W^{-1,1}} \ll 1 \qquad (e.g. \quad \varepsilon^\alpha)$$

- Define

$$r_a(t) := \|\omega(v(t)) - \pi \sum d_i \delta_{a_i(t)}\|_{W^{-1,1}}$$
$$\approx \sum |a_i(t) - \xi_i(t)|.$$

- Quantitative Γ-limit lower bound should imply

$$\pi \sum \eta^2(a_i(t)) \approx E_\varepsilon(v(t)) \geq \pi \sum \eta^2(\xi_i) - o(1)$$

and

$$\pi \sum \eta^2(\xi_i) \geq \pi \sum \eta^2(a_i) - C\,\mathrm{Lip}(\eta^2) r_a(t).$$

- Then

$$\boxed{r_a(t) \gtrsim \text{ tightness of Γ-lim lower bound .}}$$

Still more heuristics

So far

- $$r_a(t) \gtrsim \text{ tightness of } \Gamma\text{-lim lower bound .}$$

- We need
$$\frac{v_{x_i} \cdot v_{x_j}}{|\log \varepsilon|} \approx \pi \delta_{ij} \sum \delta_{\xi_i(t)}.$$
In fact this would let us control growth of $r_a(t)$.

- Note also, theorem states $r_a^\varepsilon(t) \to 0$ as $\varepsilon \to 0$.

- So we would like
$$\left\| \frac{v_{x_i}(t) \cdot v_{x_j}(t)}{|\log \varepsilon|} - \pi \delta_{ij} \sum \delta_{\xi_i(t)} \right\|_{W^{-1,1}} \le C r_a(t)$$

- however, this is not true. In fact all that holds is
$$\left\| \frac{v_{x_i}(t) \cdot v_{x_j}(t)}{|\log \varepsilon|} - \pi \delta_{ij} \sum \delta_{\xi_i(t)} \right\|_{W^{-1,1}} \le C \sqrt{r_a(t)}$$

More rigorously: Our starting point is quantitative compactness:

Lemma

Under assumptions of the theorem, there exist points $\xi_i(t)$ such that

$$\|\omega(v(t)) - \pi \sum d_i \delta_{\xi_i(t)}\|_{W^{-1,1}} \le r_\xi(t) \approx C\varepsilon^{1-r_a(t)}|\log\varepsilon|.$$

Then construct an ideal current $j^* = j^*(t)$ supported $\cup B(\xi_i(t), |\log\varepsilon|^{-1})$ such that (simplifying somewhat)

$$\|\nabla \times (j(v) - j^*)\|_{W^{-1,1}} \le Cr_\xi$$

$$\left\| \frac{j_i^* j_k^*}{|\log\varepsilon|} - \pi\delta_{ik} \sum \delta_{\xi_i(t)} \right\|_{W^{-1,1}} \le C \log\left(\frac{r_\xi(t)}{\varepsilon}\right)/|\log\varepsilon|$$

$$\|j^*\|_q \le Cr_\xi(t)^{\frac{2}{q}-1} \quad \text{for } q > 2.$$

Basic strategy: to replace $v_{x_i}v_{x_k}$ by $j_i^* j_k^*$ in identity for $\frac{d}{dt}\int \varphi\omega(v)$, and try to control errors.

main error term is $C|\log\varepsilon|^{-1}\log\frac{r_\xi(t)}{\varepsilon} \approx C|\log\varepsilon|^{-1}\log\left(\frac{\varepsilon^{1-r_a(t)}}{\varepsilon}\right) \approx r_a(t)$.

Some ingredients in the error estimates:

- split $\int \frac{\psi_{ik}}{|\log\varepsilon|}(v_{x_i} v_{x_k} - j_i^* j_k^*)$ as a sum of terms. The worst is
$$\int \frac{\psi_{ik}}{|\log\varepsilon|}(\frac{j_i(v)}{|v|} - j_i^*) j_k^* \, dx$$
- argue that $\frac{j(v)}{|v|} \approx j(v)$, and use
$$\nabla \times (j(v) - j^*) \leq C r_\xi$$
$$\nabla \cdot (\eta^2 j(v)) = \frac{1}{2}|\log\varepsilon| \partial_t [\eta^2 (|v|^2 - 1)] \qquad (2)$$
together with weighted Hodge decomposition of $j(v) - j^*$.
- integrate in t to exploit (2). So in fact we prove something like
$$r_a(t+h) - r_a(t) \leq C h r_a(t)$$
for suitable $h > 0$.
- note that some of these ingredients are not obviously connected to the Hamiltonian structure of the equation.