Variational methods for effective dynamics

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effective dynamics means:

- **Given** a nonlinear evolution equation with small or large parameter
- **one seeks** a simple description of (at least some) solutions, where
- **simple** may mean: in terms of lower-dimensional objects

relevance of the calculus of variations

- Γ-convergence is *very often* a source of inspiration
- Γ-convergence (with related estimates) is *often* an ingredient in proofs including for example for wave and Schrödinger equations
- Γ-convergence (with upgrades) can *sometimes* be the basis for proofs especially for gradient flows, *cf* lectures of Ambrosio

- In general: effective dynamics is largely a question of stability
- and calculus of variations is very relevant to stability.

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general open problems

1. Abstract framework for Γ-convergence and Hamiltonian systems.

For gradient flows, Sandier-Serfaty '04, Serfaty '11

2. For Hamiltonian systems (especially),

- Can one ever establish global-in-time results ?
- In particular, given a periodic solution of a limiting Hamiltonian system, can one find “nearby" periodic solutions for the approximating functional?

First example

Exercise

Assume that $F_\varepsilon : \mathbb{R}^2 \cong \mathbb{C} \to \mathbb{R}$ and that $F_\varepsilon \to F$ in some topology. For which topologies is it true that solutions of the ODEs

$$\dot{x}_\varepsilon = -\nabla F_\varepsilon(x_\varepsilon), \qquad x_\varepsilon(0) = x_0 \tag{1}$$

$$\ddot{x}_\varepsilon = -\nabla F_\varepsilon(x_\varepsilon), \qquad x_\varepsilon(0) = x_0, \dot{x}_\varepsilon(0) = v_0 \tag{2}$$

$$i\dot{x}_\varepsilon = -\nabla F_\varepsilon(x_\varepsilon), \qquad x_\varepsilon(0) = x_0 \tag{3}$$

converge, as $\varepsilon \to 0$, to solutions of the $\varepsilon = 0$ systems?

Note:

- for (1), “energy" decreases along trajectories: $\frac{d}{dt} F_\varepsilon(x_\varepsilon) = -|\dot{x}_\varepsilon|^2$
- for (2), “energy" is conserved: $\frac{d}{dt}[\frac{1}{2}|\dot{x}_\varepsilon|^2 + F_\varepsilon(x_\varepsilon)] = 0.$
- for (3), (a different) “energy" is conserved: $\frac{d}{dt} F_\varepsilon(x_\varepsilon) = 0.$

Another example

The above exercise is misleading in that it (probably)

- doesn't use gradient structure
- doesn't distinguish between different dynamics

more illustrative: Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a fixed smooth function, and define

$$F_\varepsilon(x, y) := g(x, y) + \varepsilon^{-p}(y - \varepsilon^q \sin(\frac{x}{\varepsilon}))^2$$

for certain $p, q > 0$.

Exercise

Show that

$$F_\varepsilon \xrightarrow{\Gamma} F_0(x, y) := \begin{cases} g(x, 0) & \text{if } y = 0 \\ +\infty & \text{if not} \end{cases}$$

Exercise

For which values of p, q do solutions of various ODEs for F_ε converge to solutions for F_0?

Note: in general Γ-convergence is far too weak to allow any conclusions about dynamics.

Exercise

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, $\phi : \mathbb{R}^n \to \mathbb{R}$ is measurable and $\mathbb{Z}^n$-periodic, with $\inf \phi = 0$. Then for any $p > 0$,

$$F_\varepsilon(x) := f(x) + \varepsilon^{-p}\phi(\frac{x}{\varepsilon}) \xrightarrow{\Gamma} f.$$

Exercise

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is positive. Let $\{x_i\}$ be a countable dense subset of $\mathbb{R}^n$, and define

$$F_\varepsilon(x) := \begin{cases} 0 & \text{if } x \in \cup_{i=1}^\infty B(x_i, 2^{-i}\varepsilon) \\ f(x) & \text{if not} \end{cases}$$

Then

$$F_\varepsilon \to F > 0 \text{ a.e. and in } L^1_{loc}, \qquad \text{but} \quad F_\varepsilon(x) \xrightarrow{\Gamma} 0.$$

For the duration of this lecture, we consider the Allen-Cahn energy

$$\boxed{F_\varepsilon(u) := \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{2\varepsilon}(u^2-1)^2\, dx} \qquad u \in H^1_{loc}(\Omega)$$

and associated evolution problems (where typically $\Omega = \mathbb{R}^n$.)

Plan

- recall Γ-convergence (for inspiration) and state corresponding wave equation result
- discuss proof
  - transform via change of variables into a stability question
  - address this using variational arguments

Theorem (Modica-Mortola '77, Modica '87,Sternberg '88)

1. (compactness) *If* $(u_\varepsilon)_{\varepsilon\in(0,1]}$ *is a sequence in* $H^1(\Omega)$ *such that*

$$F_\varepsilon(u_\varepsilon) \le C$$

then there is a subsequence that converges in L^1 *as* $\varepsilon \to 0$ *to a limit* $u \in BV(\Omega; \{\pm 1\})$*.*

2. (lower bound) *If* $(u_\varepsilon) \subset H^1(\Omega)$ *and* $u_\varepsilon \xrightarrow{L^1} u$*, then*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge F_0(u) := \begin{cases} \frac{4}{3}|Du|(\Omega) & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{if not} \end{cases}$$

3. (upper bound) *For any* $u \in L^1(\Omega)$ *there exists a sequence* $(u_\varepsilon) \subset H^1(\Omega)$ *such that*

$$u_\varepsilon \xrightarrow{L^1} u \quad \text{and } \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \le F_0(u).$$

- Informally,

$$\boxed{F_\varepsilon(\cdot) \stackrel{\Gamma}{\to} \text{“ interfacial area functional ”}}$$

- As a **corollary**: if (u_ε) is a sequence of minimizers of F_ε (for suitable boundary data....) then

$u_\varepsilon \to u \in BV(\Omega; \{\pm 1\})$ after passing to a subsequence if necessary, and

the set $\{x \in \Omega : u(x) = 1\}$ has minimal perimeter in Ω.

further (PDE) results: (*many* references omitted here.....)

- similar for nonminimizing solutions of

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon^2 - 1)u_\varepsilon = 0$$

(assuming natural energy bounds.)

- solutions of

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon^2 - 1)u_\varepsilon = \varepsilon\,\kappa$$

are related in a similar way to surfaces of Constant Mean Curvature.

- in addition,

$u_\varepsilon(x) \approx q(\frac{d(x)}{\varepsilon})$, where $d(\cdot)$ is signed distance from interface,

so that $d(\cdot)$ satisfies $|\nabla d|^2 = 1, \quad d = 0$ on Γ.

Theorem (J '11, Galvão-Sousa and J., '13)

Assume that Γ *is a smooth, compact, embedded, timelike hypersurface in* $(T_*, T^*) \times \mathbb{R}^n$*, bounding a set* $\mathcal{O}$*, and such that* $H_{mink}(\Gamma) = \kappa \in \mathbb{R}$*.*

Then there exists a sequence of solutions (u_ε) *of the wave equation*

$$\varepsilon(\partial_{tt} u_\varepsilon - \Delta u_\varepsilon) + \frac{1}{\varepsilon}(u^2 - 1)(2u - \varepsilon\kappa) = 0$$

such that

$$u_\varepsilon \to u := \begin{cases} 1 & \text{in } \mathcal{O} \\ -1 & \text{in } \mathcal{O}^c \end{cases} \qquad \text{in } L^2_{loc}((T_*, T^*) \times \mathbb{R}^n)$$

- In fact we prove more, including energy concentration around Γ, estimates of rate of convergence *etc.*
- $H_{mink} = (1 - v^2)^{-1/2}(H_{euc} - (1 - v^2)^{-1} a)$, where $v = velocity$, $a = acceleration$.
- $H_{mink} = 0 \iff$ critical point of *Minkowskian* area functional

Formal arguments and elliptic results suggest that $\boxed{u \approx q(\frac{d}{\varepsilon})}$, where

- q is the optimal 1-d profile:

$$-q'' + (q^2 - 1)q = 0, \qquad q(0) = 0, \qquad q \to \pm 1 \text{ at } \pm\infty$$

- d is the signed *Minkowski* distance function to Γ, *i.e.*

$$-(\partial_t d)^2 + |\nabla d|^2 = 1, \qquad d = 0 \text{ on } \Gamma,$$

Note that q minimizes

$$v \mapsto \int_{\mathbb{R}} \frac{1}{2}(v')^2 + \frac{1}{2}(1 - v^2)^2 dr$$

among functions such that $v(r) \to \pm 1$ as $r \to \pm\infty$.

Plan:

- consider $1d$ case (with $\kappa \neq 0$) for simplicity
- Let $r_0 = \kappa^{-1}$. Then

$$\Gamma = \{r_0(\sinh\theta, \cosh\theta) : \theta \in \mathbb{R}\} := \{(t,x) : x^2 - t^2 = r_0^2\}$$

- change to Minkowskian polar coordinates

$$(r\cosh\theta, r\sinh\theta) = (x,t), \qquad r > 0, \theta \in \mathbb{R}$$

Then $\boxed{\theta \approx \text{“\textit{time}”}}$ and $\boxed{r - r_0 = d = \text{Minkowski distance to } \Gamma}$

- then hope to show that

$$\boxed{u(x,t) = v(r,\theta) \approx q(\frac{d}{\varepsilon}) = q(\frac{r - r_0}{\varepsilon}).}$$

- in fact we will concoct a functional ζ such that

$$\eta(v(\cdot,\theta)) \text{ small } \Rightarrow v \approx q(\frac{r-r_0}{\varepsilon}),$$

$$\frac{d}{d\theta}\eta(v(\cdot,\theta)) \approx 0 \text{ if } v \approx q(\frac{r-r_0}{\varepsilon}).$$

- **change variables** to study $\varepsilon(v_{\theta\theta} - v_{rr} - \frac{1}{r}v_r) + \frac{1}{\varepsilon}(v^2-1)(v-2\varepsilon\kappa) = 0$.
- **data1**: take $v(r,0) = -1, v_\theta(r,0) = 0$ if $r < R^{-1}$, $v(r,0) = 1, v_\theta(r,0) = 0$ if $r > R$.

  Then: $v(r,\theta) = -1$ near $r = 0$, and $v(r,\theta) = 1$ for $r > Re^{|\theta|}$. $(\star)$
- **data2**: take $v(r,0) \approx \tanh(\frac{r-r_0}{\varepsilon})$, for $r_0 = \kappa^{-1}$. Also take $v_\theta(0,r) \approx 0$.

  motivation: $q = \tanh(\frac{r-r_0}{\varepsilon})$ solves $\varepsilon(-q_{rr} - \frac{1}{r}q_r) + f_\varepsilon(v) = 0$. Close to PDE with $v_\theta \equiv 0$.
- **pseudo-energy**: define $e_\varepsilon(v) = \frac{\varepsilon}{2}(\frac{1}{r^2}v_\theta^2 + v_r^2) + \frac{1}{2\varepsilon}(v^2-1)^2$.

  compute: $\frac{d}{d\theta}e_\varepsilon(v) = \varepsilon(v_\theta v_r)_\theta + \underbrace{\varepsilon r^{-1}v_\theta v_r(1-\kappa r)}_{\text{Term 1}} + \underbrace{\varepsilon\kappa v_\theta(v_r - \varepsilon^{-1}(1-v^2))}_{\text{Term 2}}$.
- **optimality of** q: $(\star)$ implies $\int_0^\infty e_\varepsilon(v)dr \ge c_0$, equality iff $v_r - \varepsilon^{-1}(1-v^2) = 0$ iff $v = q(r-a)$.
- **define**: $\boxed{\eta_1(\theta) = \int (1+(r-r_0)^2)e_\varepsilon(v)(r,\theta)dr - c_0.}$ Then **data2** implies $\eta_1(0) \le C\varepsilon^2$.

  then: $\boxed{\eta_1(\theta) \ge \eta_2(\theta) = \int \frac{\varepsilon}{2}\frac{v_\theta^2}{r^2} + (r-r_0)^2(\frac{\varepsilon}{2}v_r^2 + \frac{1}{2\varepsilon}(v^2-1)^2)dr}$ if $(\star)$ holds.
- **estimate** $\boxed{\frac{d}{d\theta}\eta_1(\theta) \le Ce^{C|\theta|}\eta_2(\theta) \le Ce^{C|\theta|}\eta_1(\theta)}$ if $(\star)$ holds. (C depends only on R.)
- **conclude that** $\boxed{\eta_1(\theta) \le Ce^{Ce^{C|\theta|}}\varepsilon^2}$.

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