Gradient flows in the framework of (Cartesian) Currents

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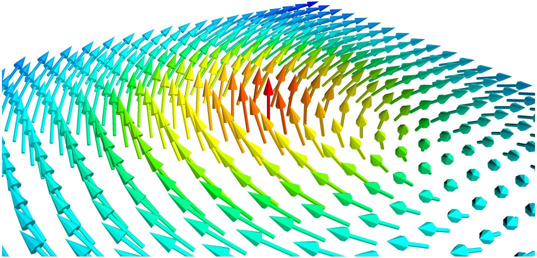
Vortices as singularities



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What happens if we penalize the third component?

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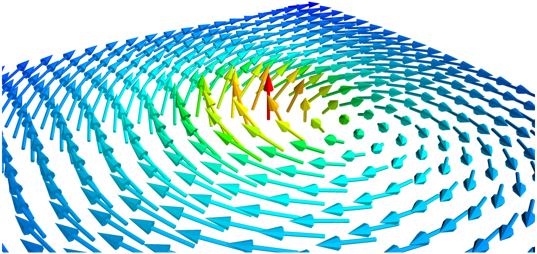

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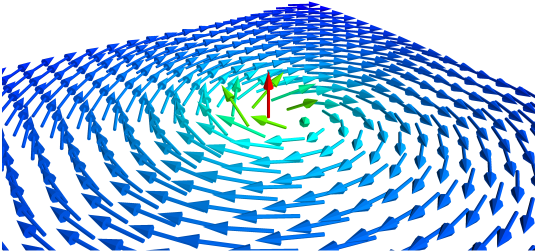

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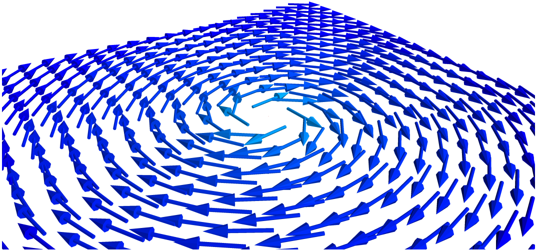

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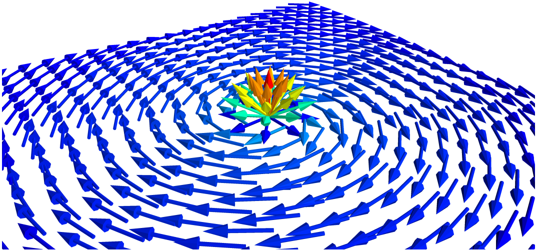

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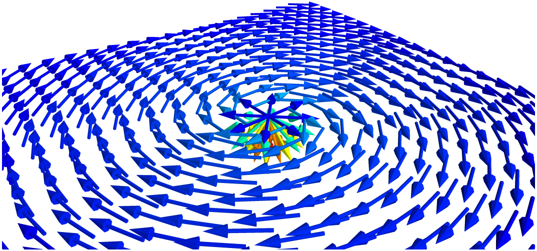

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Bubbling and Vertical parts

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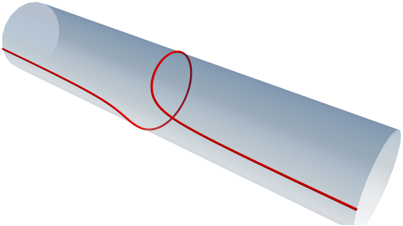

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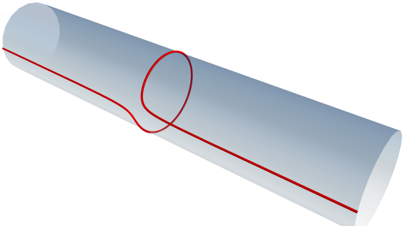

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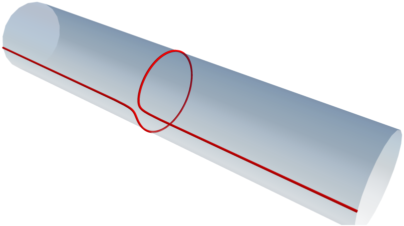

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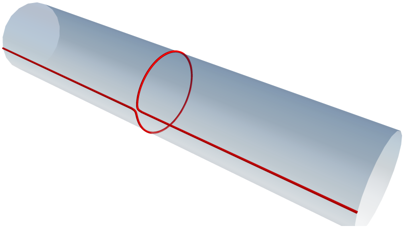

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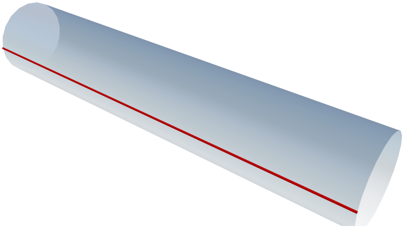

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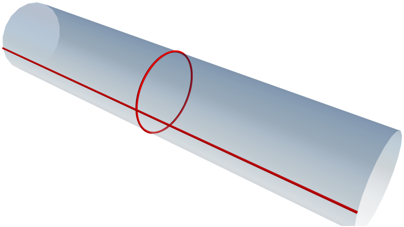

Cartesian currents

Approach due to Giaquinta, Modica, Souček ('89):

- Consider graphs of (nice enough) functions $\Omega \subset \mathbb{R}^n \to \mathcal{M}$ as rectifiable n-current in $\Omega \times \mathcal{M}$.
- Cartesian Currents $\approx$ closure the class of graphs in the topology of currents

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Reminder (de Rham ('55), Federer & Fleming ('60)):

- k-Currents $\approx$ dual space of compactly supported smooth differential k-forms (approach similar to distributions)
- Rectifiable k-currents $\approx$ countable unions of orientable manifolds with integer multiplicity

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Some features:

- Cartesian Currents usually consist of flat "graph" part and "vertical" singularities
- Cart. Currents are boundaryless (boundary in $\partial(\Omega \times \mathcal{M})$ does not count)

Why gradient flows

- Large class of similar problems
- Many have some sort of singularities
- Canonical example: Harmonic map heat flow
- Good abstract approach available (s. book by Ambrosio, Gigli, Savaré)

Minimizing movements (de Giorgi)

Ingredients

- Set of admissible Currents $\mathcal{A}$
- Metric $d(.,.)$
- Energy $E(.)$

Implicit Euler iteration

$$S_{k+1}^{(h)} := \arg\min \left\{ \frac{1}{2h} d\left(S, S_k^{(h)}\right)^2 + E(S) \middle| S \in \mathcal{A} \right\}.$$

Then for $h \to 0$ the limit $S(t) = \lim_{h\to 0} S_{k/h}^{(h)}$ should converge to a solution to the gradient flow

$$\frac{\partial}{\partial t} S + \nabla_d E(S) = 0$$

Convergence theorem (K. 2014)

Assume we have some closed class of cartesian currents $\mathcal{A}$ for which

i) d^2 and E lower semi-continuous

ii) E bounded from below

iii) The mass of currents with bounded energy is bounded

iv) $\dot{\mathcal{F}}(S-T) \leq c \cdot d(S,T)$ for some c and all S, T of bounded energy

Then the minimizing movements iteration is well defined and converges (up to a subsequence) on any time interval $[0,\tau]$ to an $k+1$ space-time current A s.t. the approximations $S^{(h)}(r)$ converge to the slices $\langle A, t < r\rangle$.

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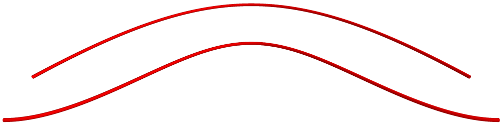
The boundary free case: a homogeneous Flat norm



Reminder: Flat norm

$$\mathcal{F}(S - T) := \sup\{(S - T)(\omega) : \|\omega\| \leq 1 \wedge \|d\omega\| \leq 1\}$$
$$= \inf\{M(A) + M(B) : S - T = \partial A + B\}$$

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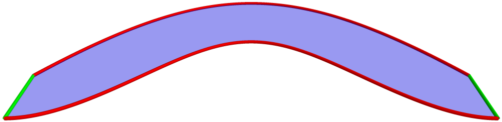

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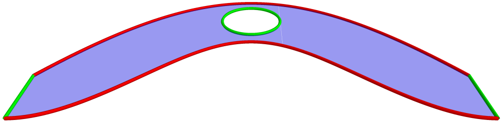

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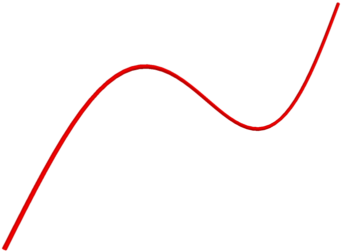
Preserves boundary and topology, i.e. $\dot{\mathcal{F}}(S - T)$ is infinite for topologically different currents, suitable for Cartesian Currents.

Sketch of proof



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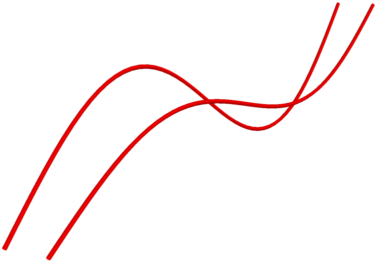

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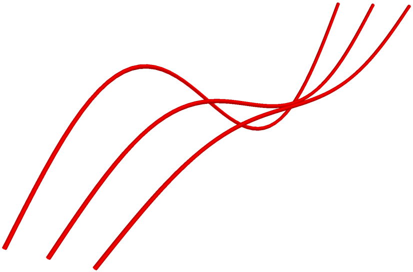

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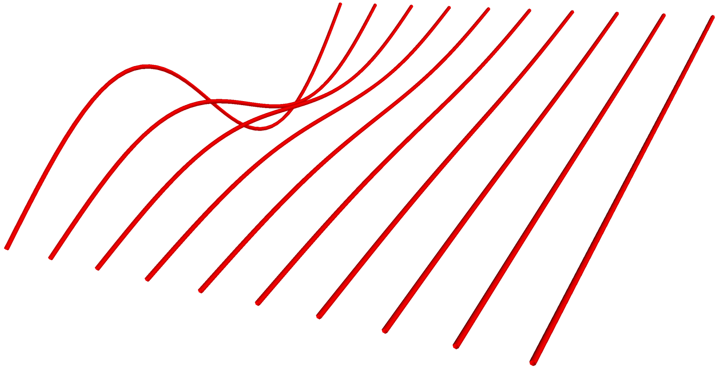

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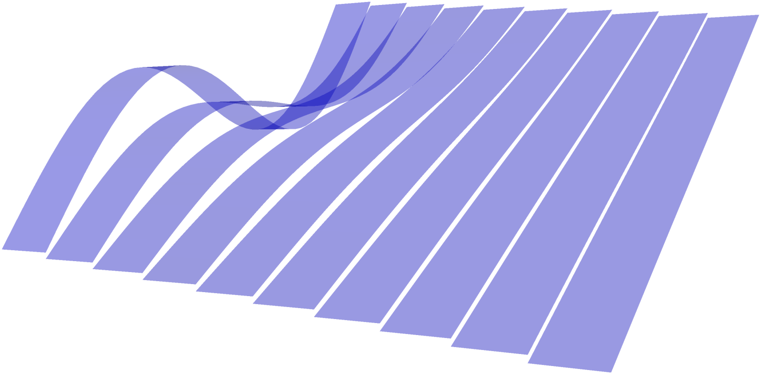

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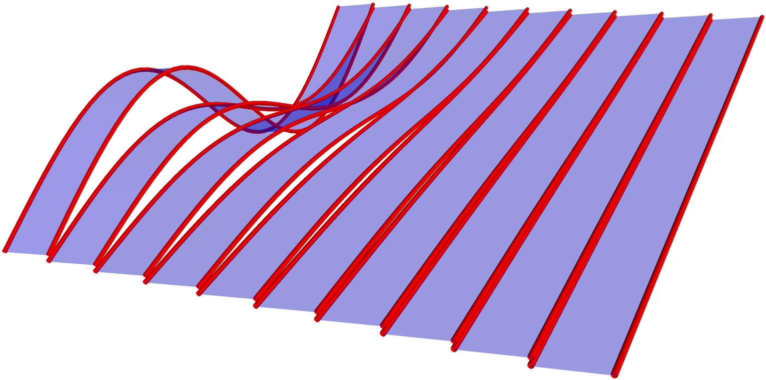

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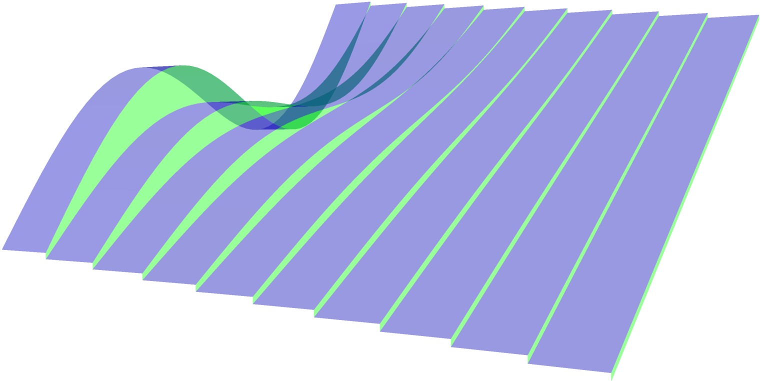

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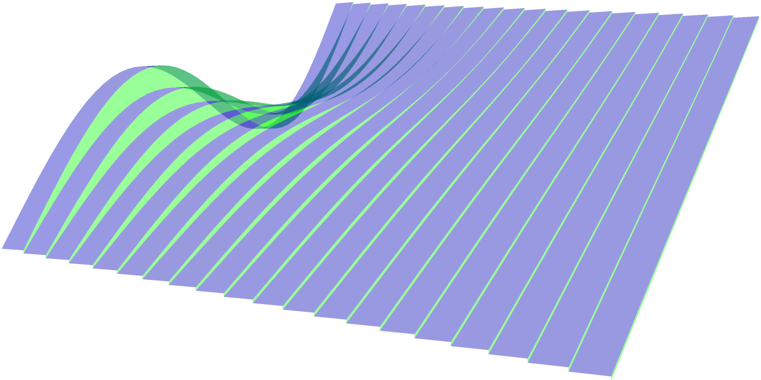

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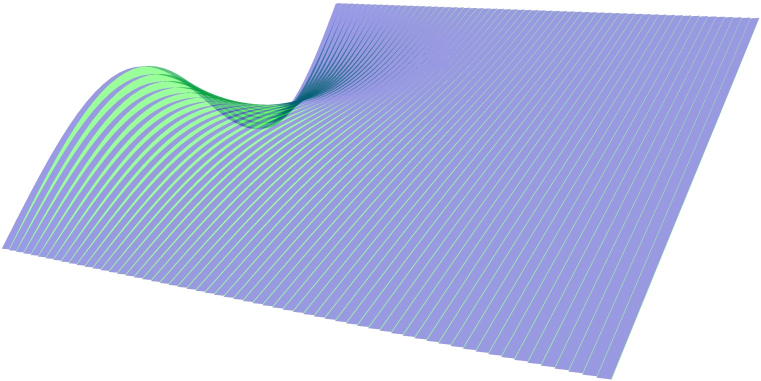

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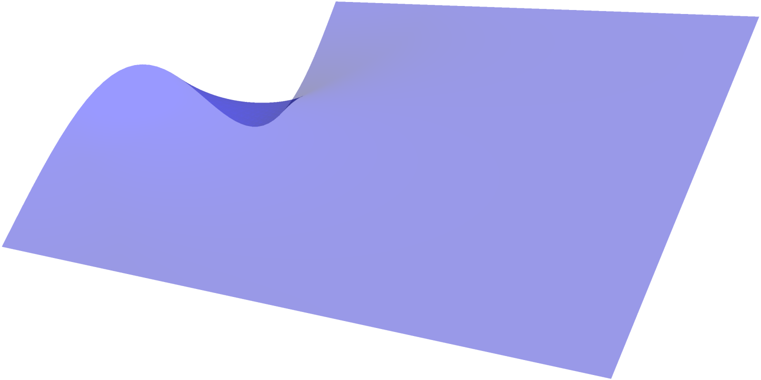
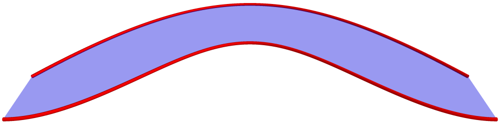


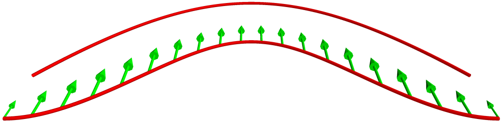
Finding an L^2 norm that isn't the L^2 norm



What is a good metric?

- For many problems, candidate metric needs to behave similar to L^2 distance
- Known metric: (homogeneous) Flat norm
- However: Behaves more like L^1 distance

Approaching the problem from a different direction: Wasserstein distance



$$W_2(\nu_0, \nu_1)^2 = \inf_{\pi \in \Pi(\nu_0, \nu_1)} \int dist(x, y)^2 \, d\pi(x, y)$$

- Problem: We need to preserve multiplicity
- However: Wasserstein distance preserves mass instead
- Different interpretation: Treat distance as moving the current by a vector field

Wasserstein distance: A geometric viewpoint

- Conservation of mass formula:

$$\partial_t \mu + \nabla \cdot (v\mu) = 0$$

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$$\partial_t \mu(\phi) = \mu(v \cdot \nabla \phi) \quad \forall \phi$$

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- Measure $\approx$ 0-current:

$$\partial_t T(\omega) = T(i_v d\omega) \quad \forall \omega$$

Contraction: $(i_v \omega)(w_1, ..., w_n) = \omega(v, w_1, ..., w_n)$

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- Since $i_v \omega = 0$ for 0-forms

$$\partial_t T(\omega) = T(\mathcal{L}_v \omega) \quad \forall \omega$$

Cartan's Magic Formula: $\mathcal{L}_v(\omega) = i_v d\omega + d i_v \omega$

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This formula generalises to currents of higher order, conservation of mass changes to conservation of multiplicity.

Wasserstein distance on rectifiable currents

Classical Wasserstein distance

- Norm in "tangential space":

$$\|s\|_{\mu,p}^{p} := \inf \left\{ \int |v|^p \, d\mu \,\middle|\, s + \nabla \cdot (v\mu) = 0 \right\}$$

- Wasserstein distance as length of minimal curve:

$$\mathcal{W}_p(\mu_0, \mu_1)^p := \inf_{\mu} \left\{ \int_0^1 \left\| \frac{\partial \mu(t)}{\partial t} \right\|_{\mu(t),p}^{p} dt \,\middle|\, \mu(i) = \mu_i, i \in \{0,1\} \right\}$$

Wasserstein distance on rectifiable currents

Adaptation to currents

- Norm in "tangential space":

$$\|S\|_{T,p}^{p} := \inf \left\{ \int |v|^p \, d\, \|T\| \,\middle|\, S + \mathcal{L}_v T = 0 \right\}$$

- Wasserstein distance as length of minimal curve:

$$\mathcal{W}_p(T_0, T_1) := \inf_S \left\{ \int_0^1 \left\| \frac{\partial T(t)}{\partial t} \right\|_{T(t),p}^{p} dt \,\middle|\, S(0) = T_0, S(1) = T_1 \right\}$$

Wasserstein distance on rectifiable currents

Adaptation to cartesian currents

- Norm in "tangential space":

$$\|S\|_{T,p}^{p} := \inf \left\{ \int |(\pi_\Omega)_* v|^p \, d\, \|(\pi_\Omega)_* T\| \,\middle|\, S + \mathcal{L}_{(0,v)} T = 0 \right\}$$

- Wasserstein distance as length of minimal curve:

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Some nice observations

- In general for smaller distances the vertical version is equivalent to the L^p norm.
- However: Generalized Wasserstein distance respects the topology
- For $p = 1$: generalised Wasserstein $\equiv$ homogeneous flat norm
- For $p = 1, n = 0$: sup/inf duality coincides with the Kantorovich-Rubinstein duality
- Variant with boundary possible