The continuum limit of distributed dislocations

Cy Maor

Institute of Mathematics, Hebrew University

Conference on non-linearity, transport, physics, and patterns

Fields Institute, October 2014

Different Models for Dislocations

Different Models for Dislocations

- A single dislocation

Different Models for Dislocations

- A single dislocation
  - Volterra (~1900)

Different Models for Dislocations

- A single dislocation
  - Volterra (~1900)
  - Riemannian manifold with singularities.

Different Models for Dislocations

- A single dislocation
  - Volterra (~1900)
  - Riemannian manifold with singularities.
  - Burgers vector

Different Models for Dislocations

- A single dislocation
  - Volterra (~1900)
  - Riemannian manifold with singularities.
  - Burgers vector
- Distributed dislocations

Different Models for Dislocations

- A single dislocation
  - Volterra (~1900)
  - Riemannian manifold with singularities.
  - Burgers vector
- Distributed dislocations
  - Nye, Bilby, etc. (~1950)

Different Models for Dislocations

- A single dislocation
  - Volterra (~1900)
  - Riemannian manifold with singularities.
  - Burgers vector

- Distributed dislocations
  - Nye, Bilby, etc. (~1950)
  - Smooth manifold with a torsion field (=Burgers vector density).

Different Models for Dislocations

- A single dislocation
  - Volterra (~1900)
  - Riemannian manifold with singularities.
  - Burgers vector
- Distributed dislocations
  - Nye, Bilby, etc. (~1950)
  - Smooth manifold with a torsion field (=Burgers vector density).

How to bridge between the descriptions? What kind of homogenization process yields a torsion field from singularities?

Different Models for Dislocations

- A single dislocation
  - Volterra (~1900)
  - Riemannian manifold with singularities.
  - Burgers vector

- Distributed dislocations
  - Nye, Bilby, etc. (~1950)
  - Smooth manifold with a torsion field (=Burgers vector density).

How to bridge between the descriptions? What kind of homogenization process yields a torsion field from singularities?

A new limit concept in differential geometry!

Continuum Limit of Dislocations

Continuum Limit of Dislocations

- Overview:

Continuum Limit of Dislocations

- Overview:
  - What is an edge-dislocation?

Continuum Limit of Dislocations

- Overview:
  - What is an edge-dislocation?
  - Construction of manifolds with many dislocations.

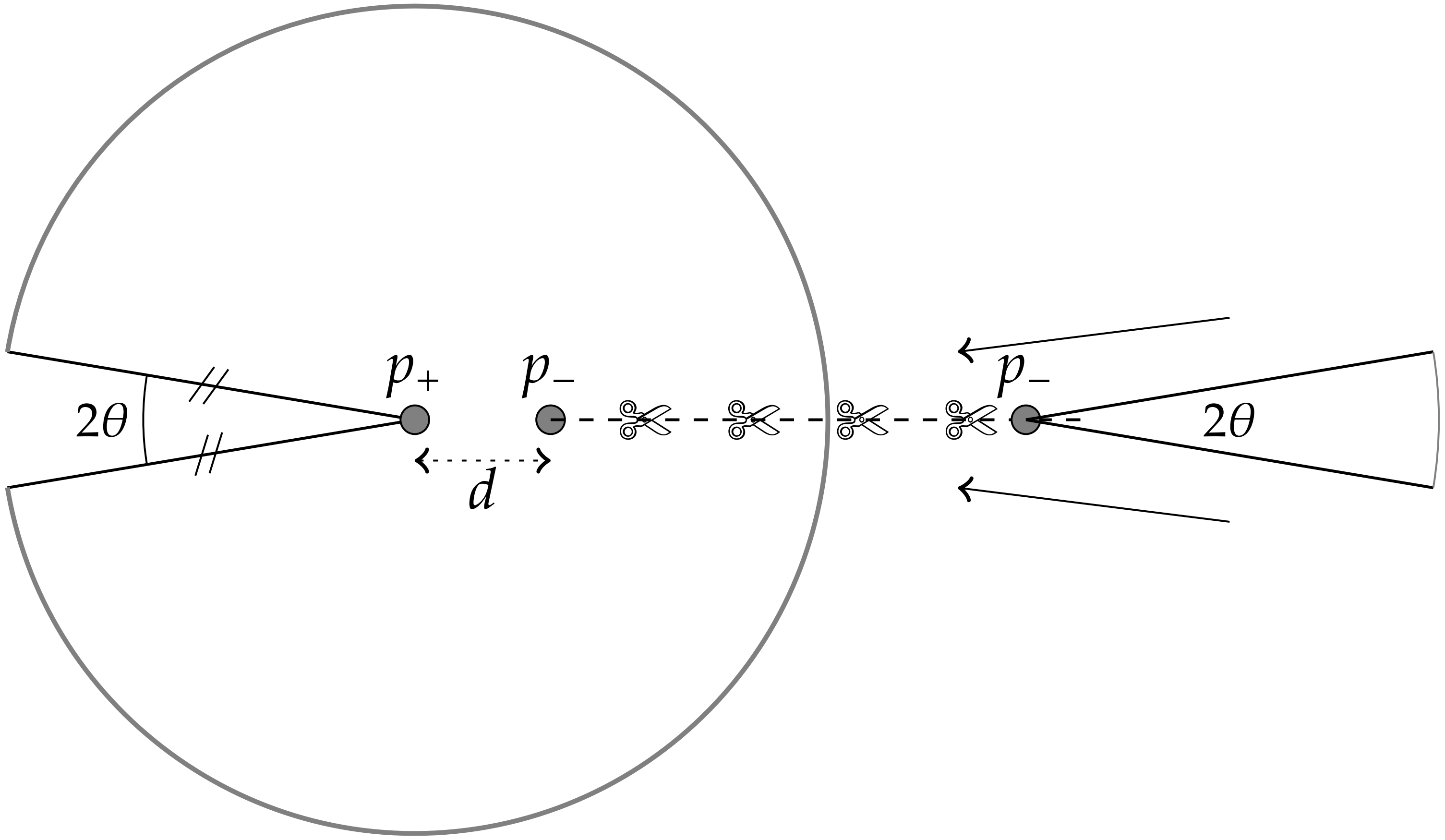
Continuum Limit of Dislocations

- Overview:
  - What is an edge-dislocation?
  - Construction of manifolds with many dislocations.
  - Dislocations become denser — what does converge?

Continuum Limit of Dislocations

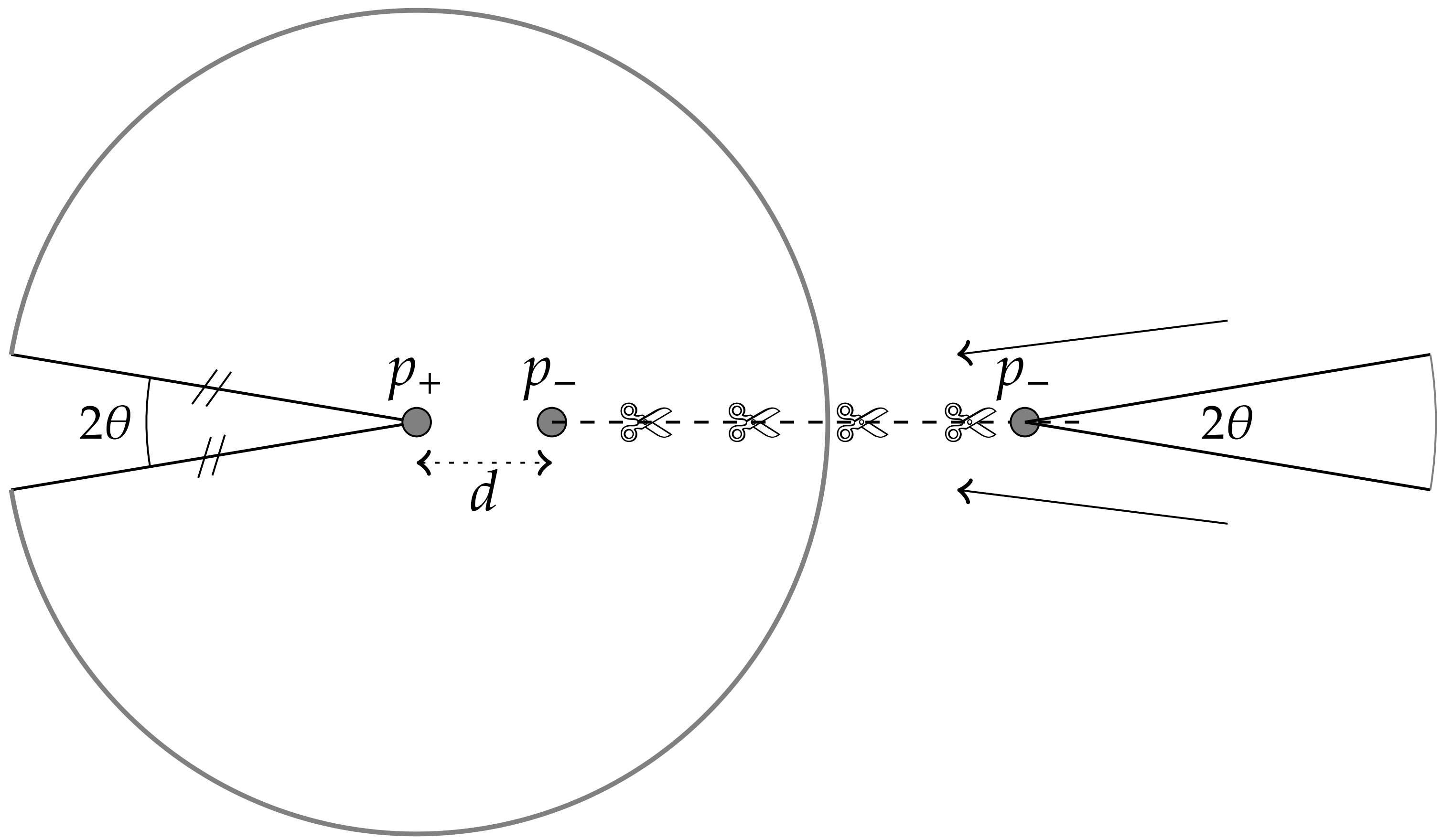
- Overview:
  - What is an edge-dislocation?
  - Construction of manifolds with many dislocations.
  - Dislocations become denser — what does converge?
  - Connection to the classical model of distributed dislocations.

An edge-dislocation



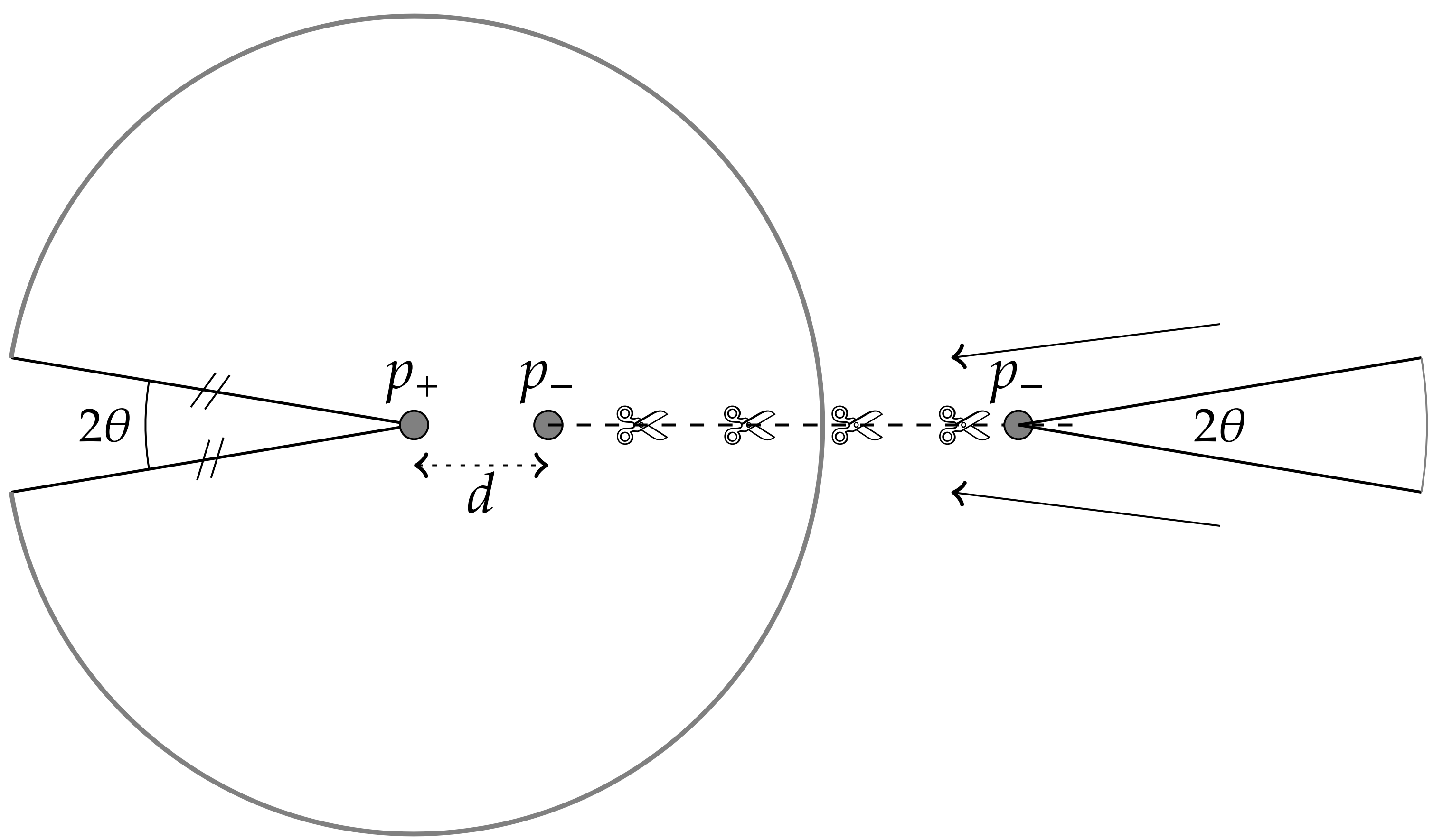
An edge-dislocation

- Remove a sector of angle *2θ*, and glue the edges (a cone).



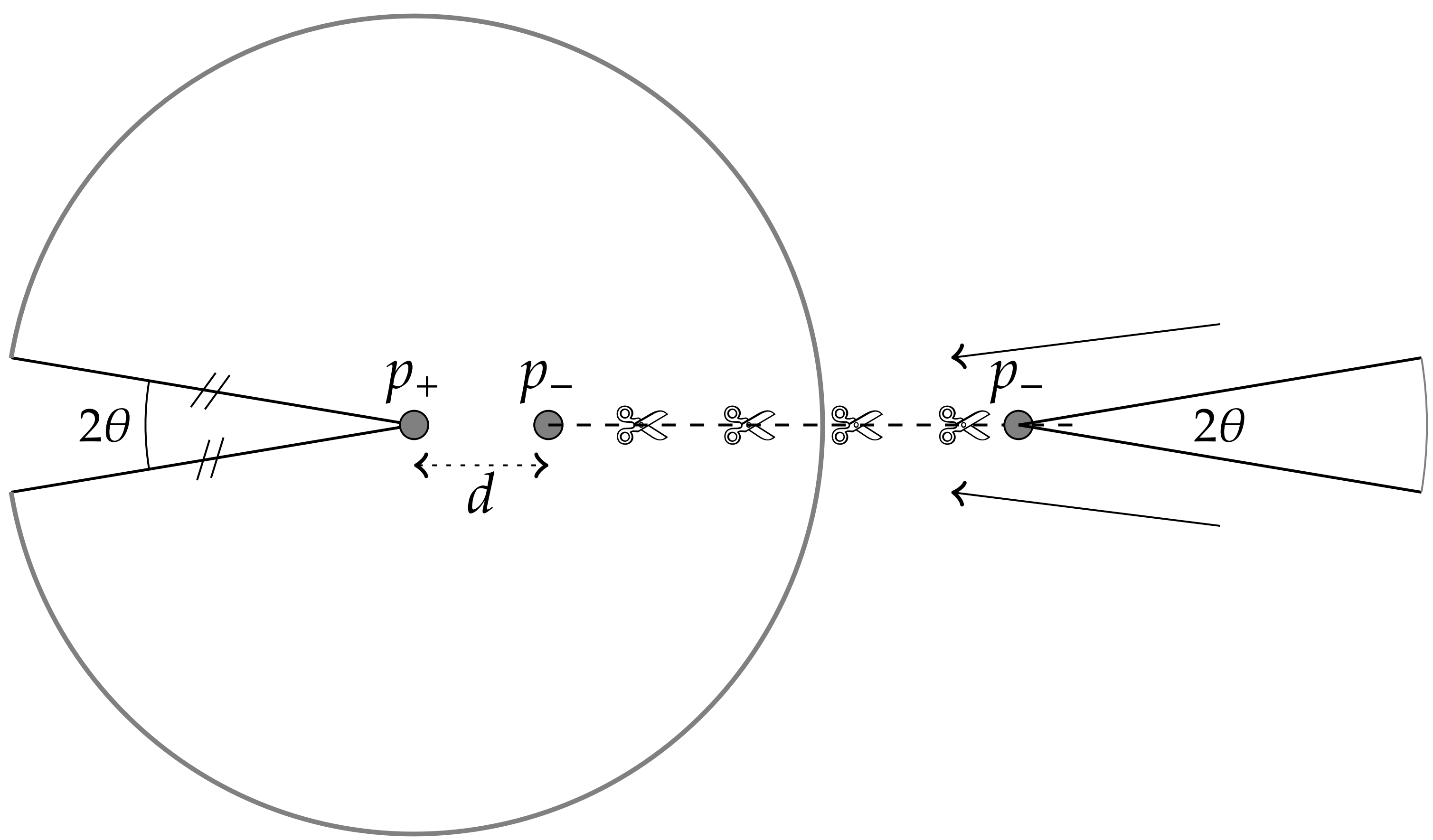
An edge-dislocation

- Remove a sector of angle 2θ, and glue the edges (a cone).
- Choose a point at distance d from the tip of the cone, cut a ray from it, and insert the sector into the cut.

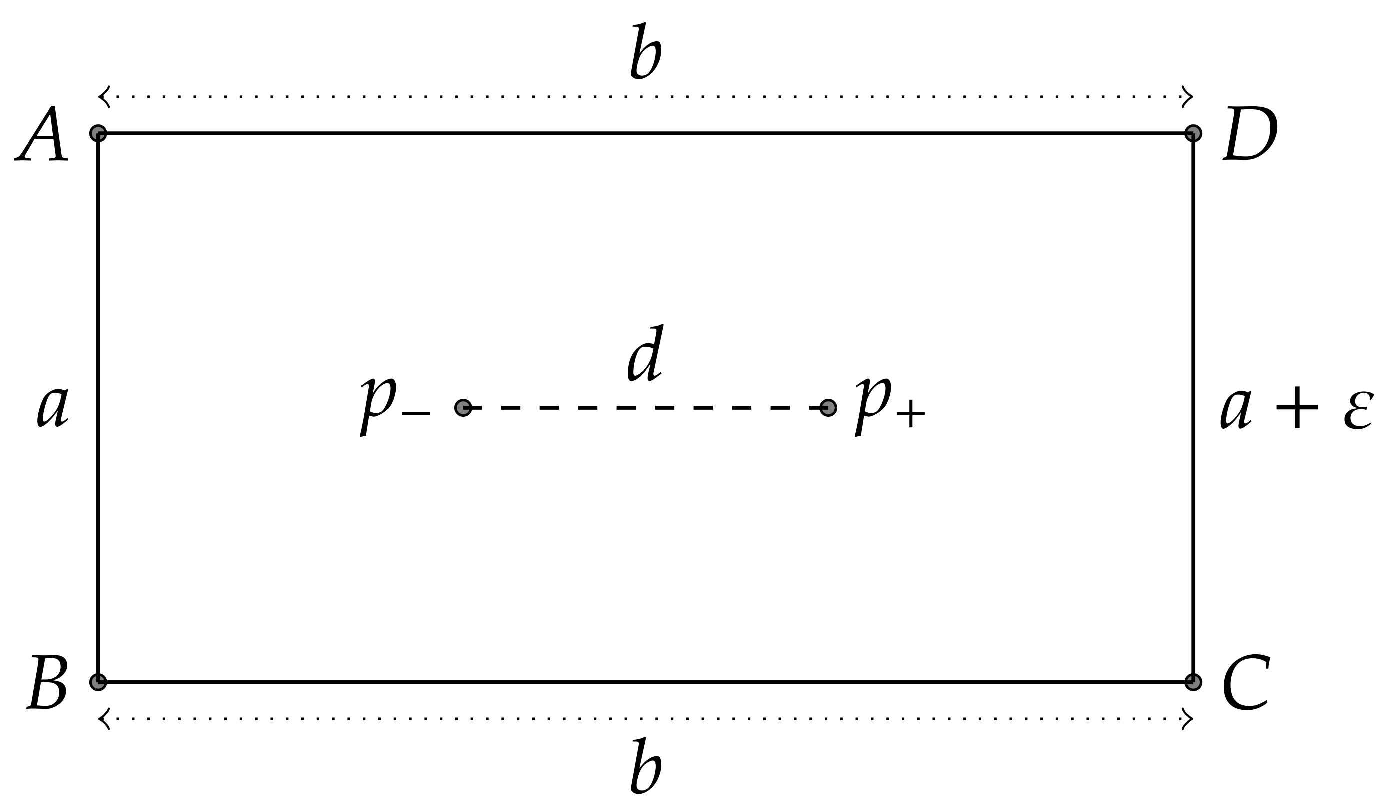


An edge-dislocation

- Remove a sector of angle *2θ*, and glue the edges (a cone).
- Choose a point at distance *d* from the tip of the cone, cut a ray from it, and insert the sector into the cut.
- A simply connected metric space, a smooth manifold outside the **dislocation line** $[p_-,p_+]$.

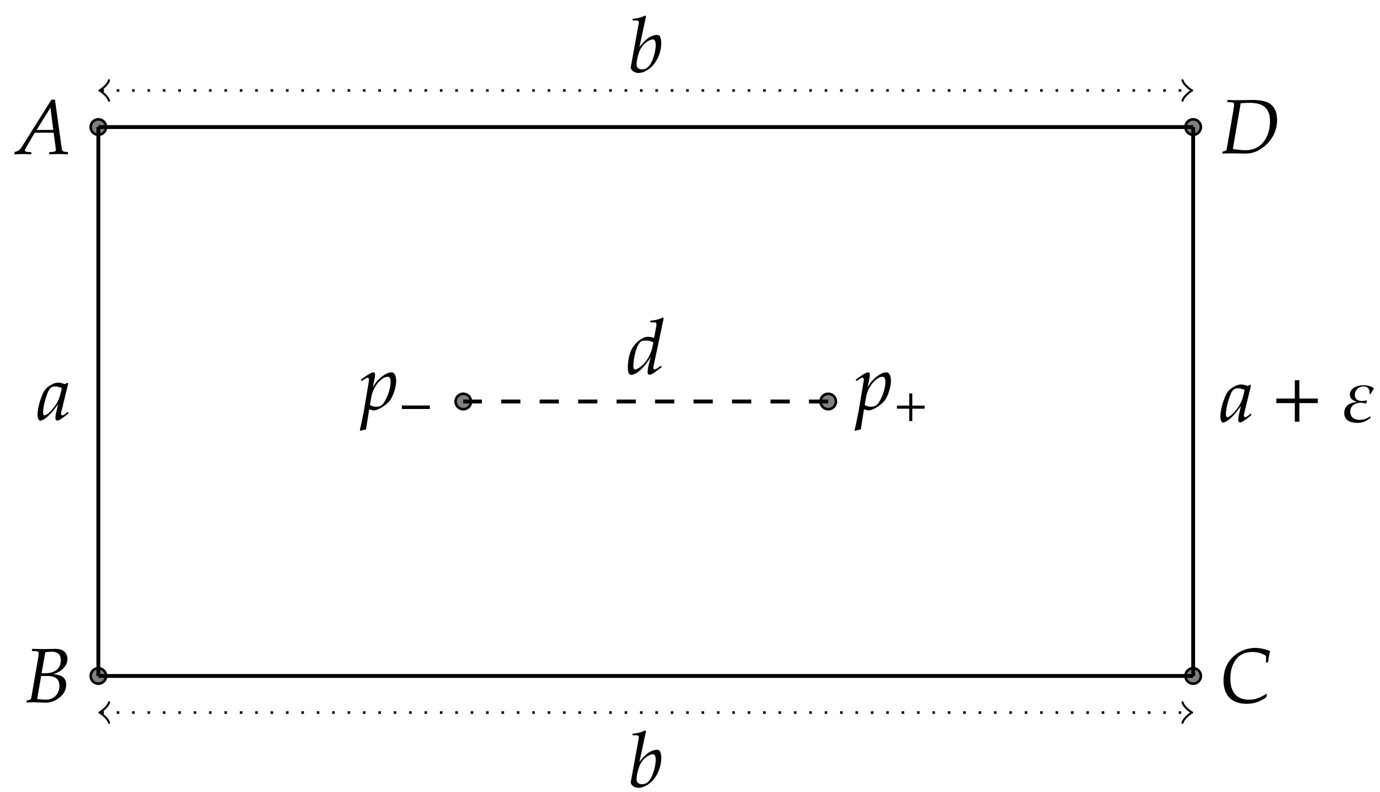


The building block



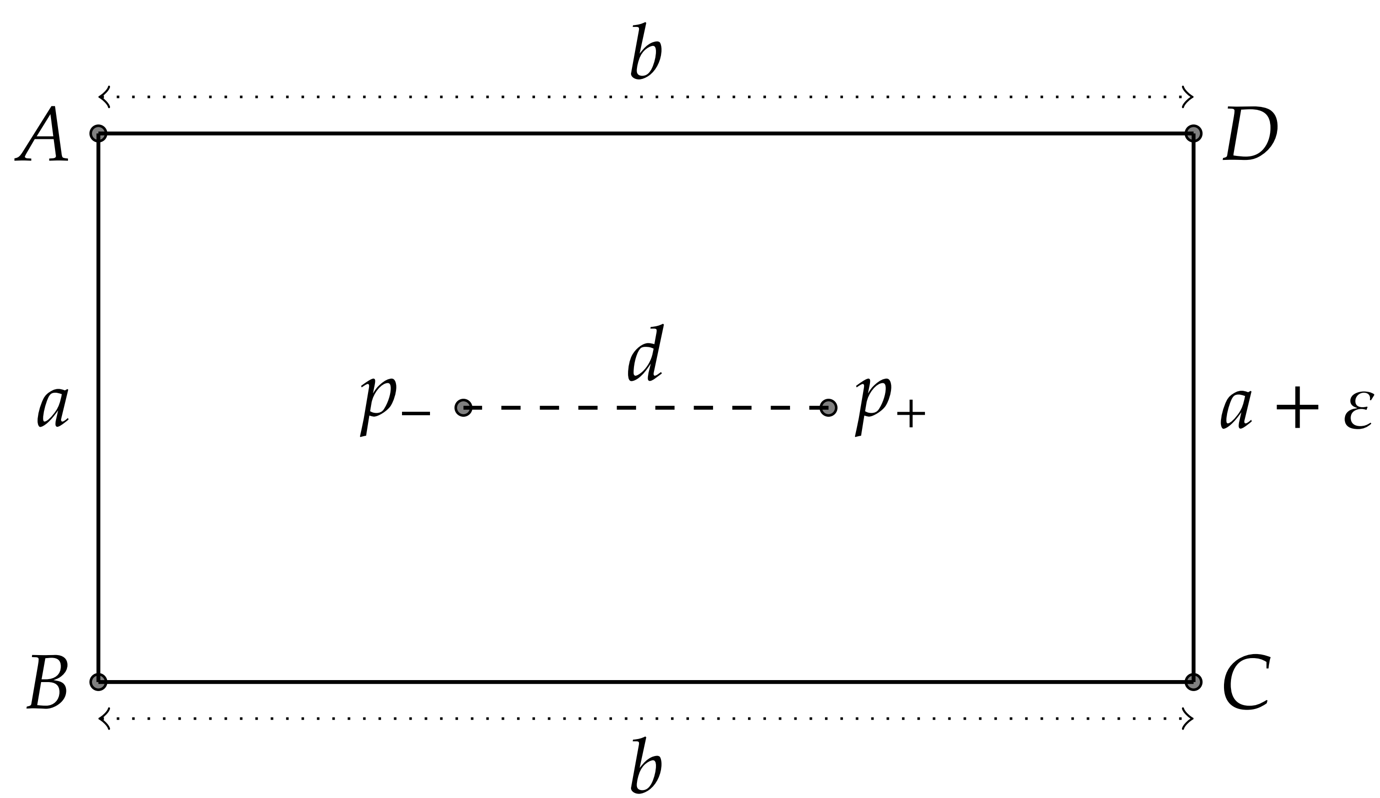
The building block

- Encircle the dislocation line with four straight lines with right angles between them, obtaining a “rectangle”.

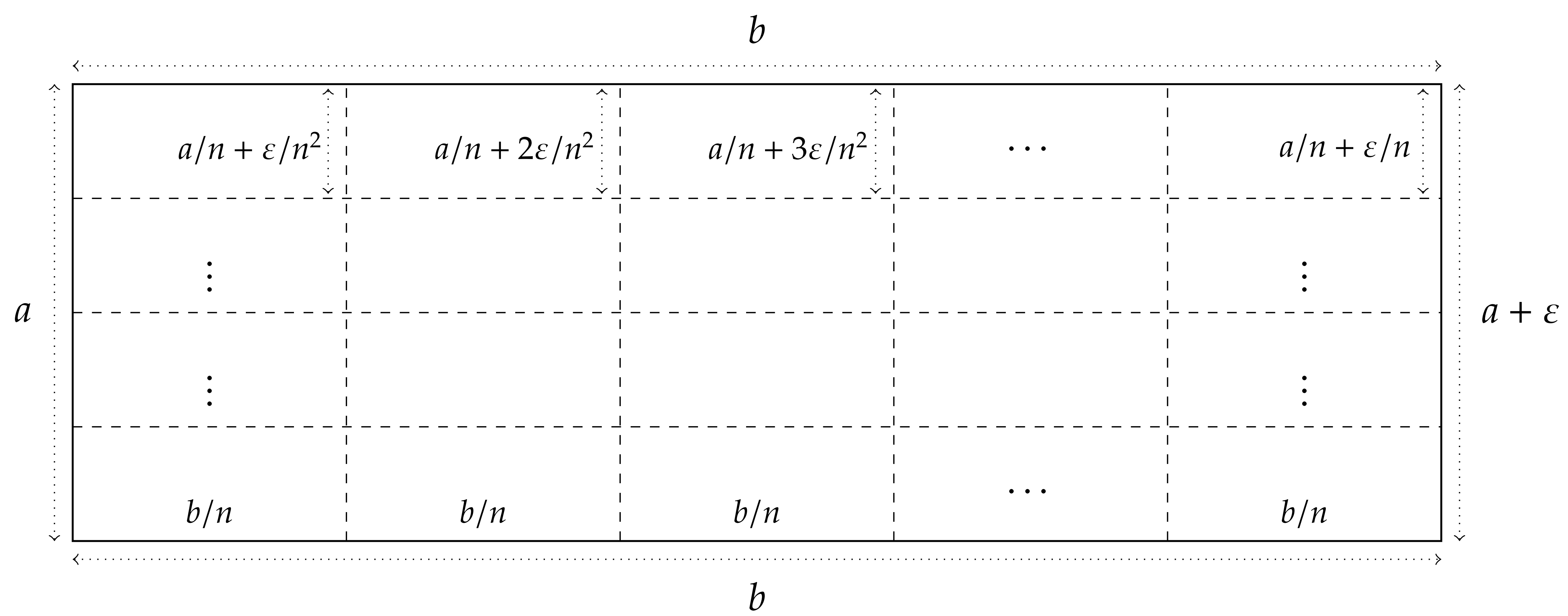


The building block

- Encircle the dislocation line with four straight lines with right angles between them, obtaining a "rectangle".
- Denote the lengths of these lines by *a*, *b*, *b*, and $a+\varepsilon$, where $\varepsilon = 2d\sin\theta$ is the **dislocation magnitude**.

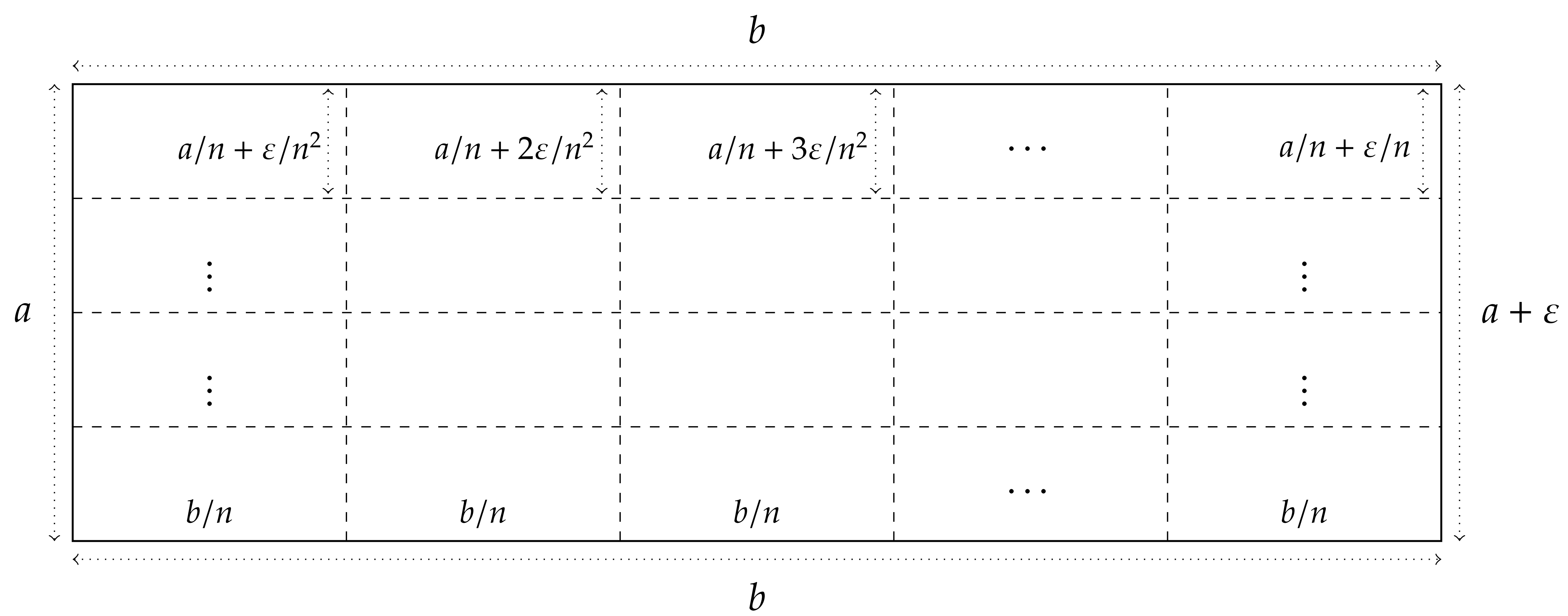


Manifolds with many dislocations



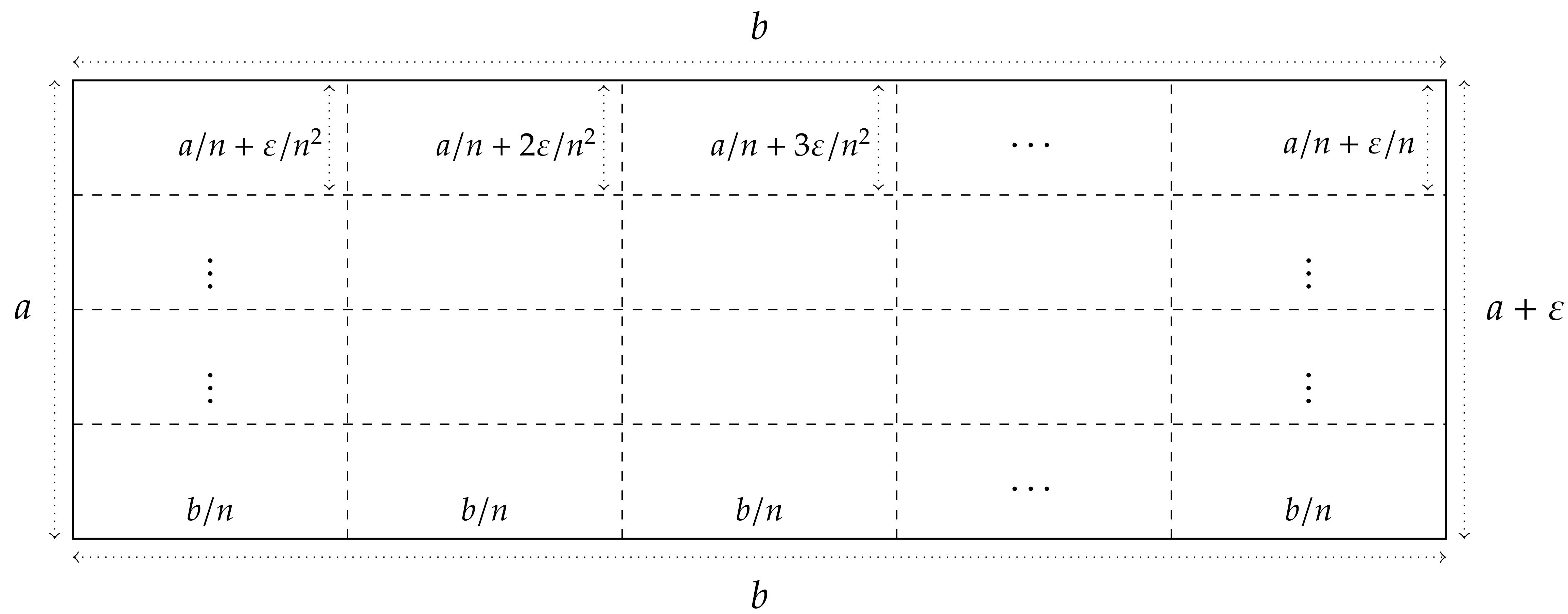
Manifolds with many dislocations

- Glue together n^2 building blocks, such that:



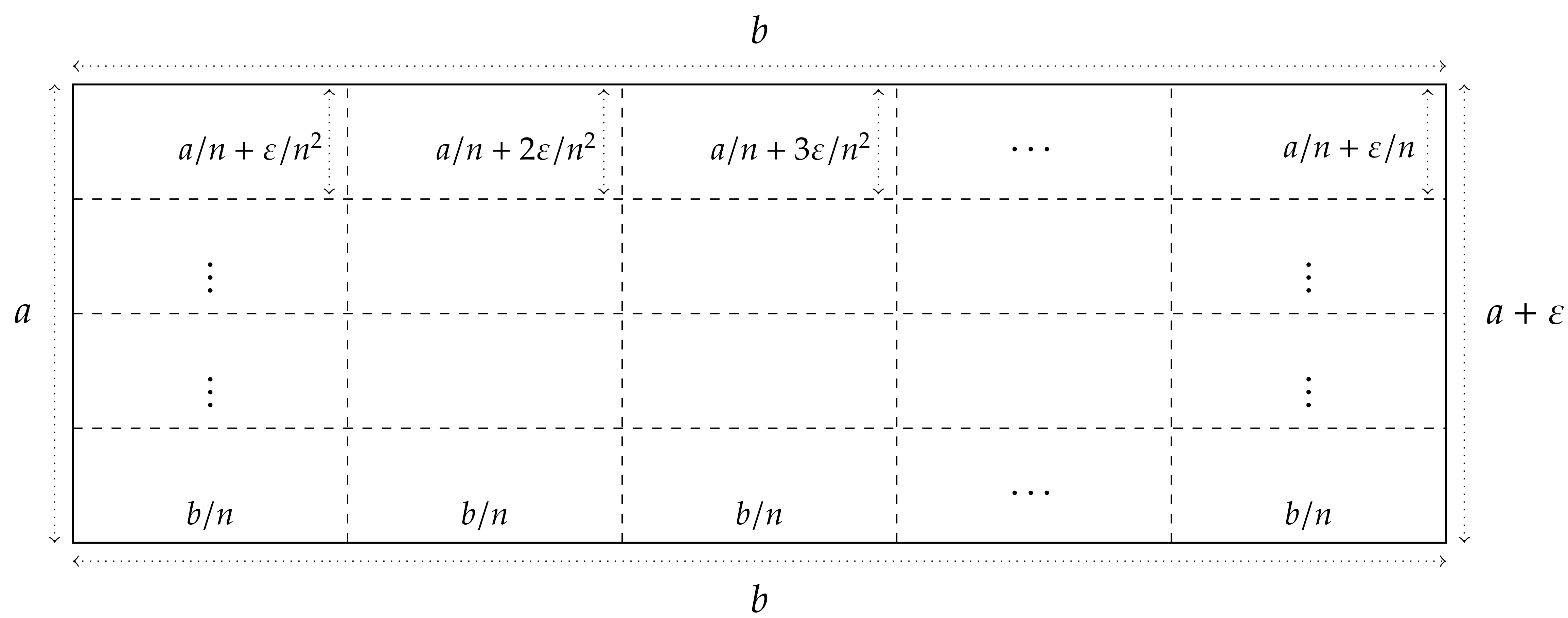
Manifolds with many dislocations

- Glue together n^2 building blocks, such that:
  - Each with the same cone angle 2θ and with dislocation magnitude ε/n^2.



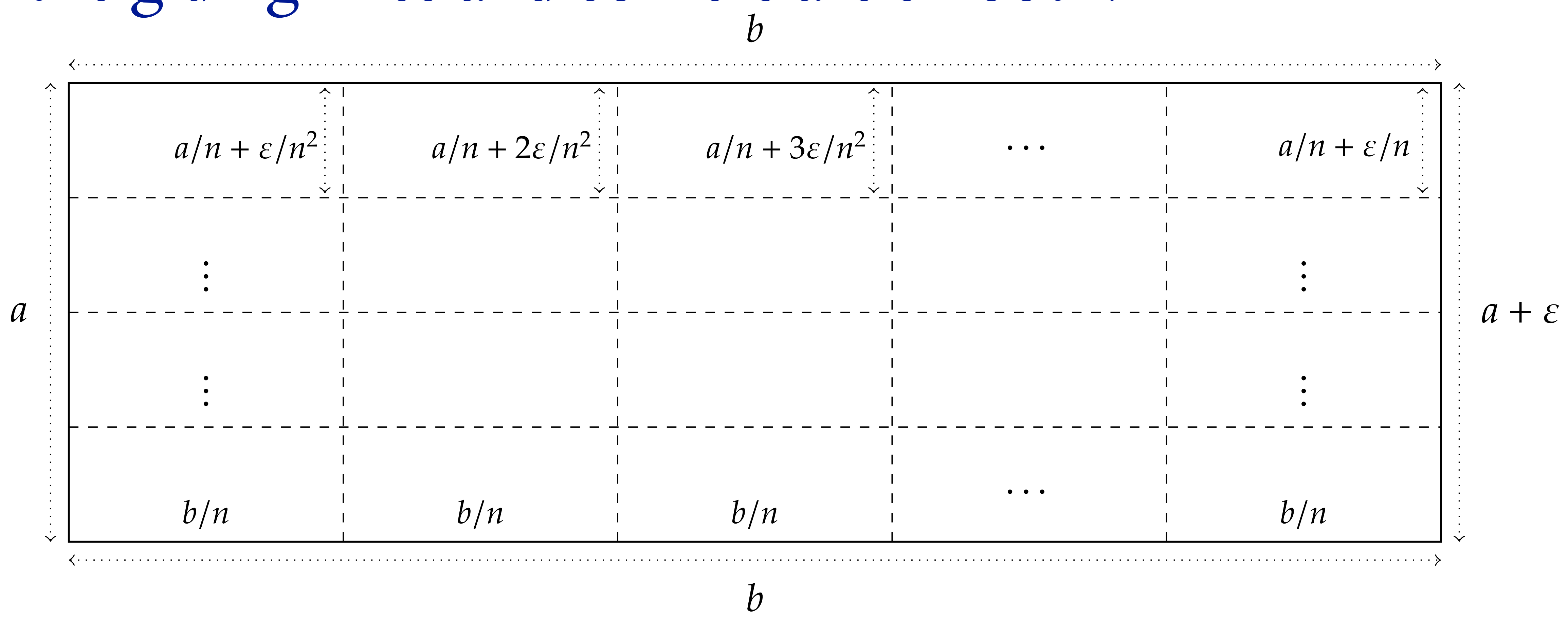
Manifolds with many dislocations

- Glue together n² building blocks, such that:
  - Each with the same cone angle 2θ and with dislocation magnitude ε/n^2.
  - The boundary consists of straight lines of lengths a, b, b, and $a+\varepsilon$.



Manifolds with many dislocations

- Glue together n^2 building blocks, such that:
  - Each with the same cone angle 2θ and with dislocation magnitude ε/n^2.
  - The boundary consists of straight lines of lengths a, b, b, and $a+\varepsilon$.
- The rectangular properties of the blocks ensure us that the gluing lines and corners are smooth.



Metric Convergence

Metric Convergence

How do these manifolds M_n look like when $n \to \infty$?

Metric Convergence

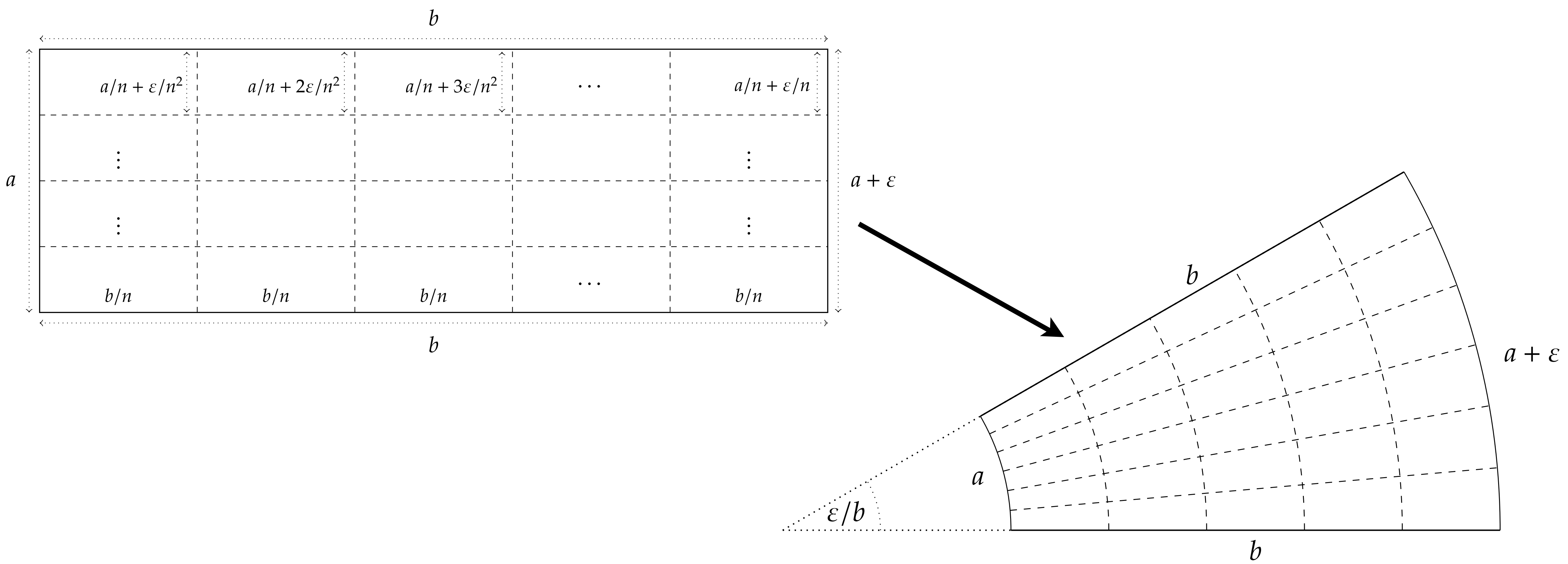
How do these manifolds M_n look like when $n \to \infty$?

Theorem: The sequence M_n converges in the *Gromov-Hausdorff* sense, to M, a sector of a flat annulus whose boundary consists of curves of lengths a, b, b, and $a+\varepsilon$.

Metric Convergence

How do these manifolds M_n look like when $n \to \infty$?

<u>Theorem:</u> The sequence M_n converges in the *Gromov-Hausdorff* sense, to M, a sector of a flat annulus whose boundary consists of curves of lengths a, b, b, and $a+\varepsilon$.



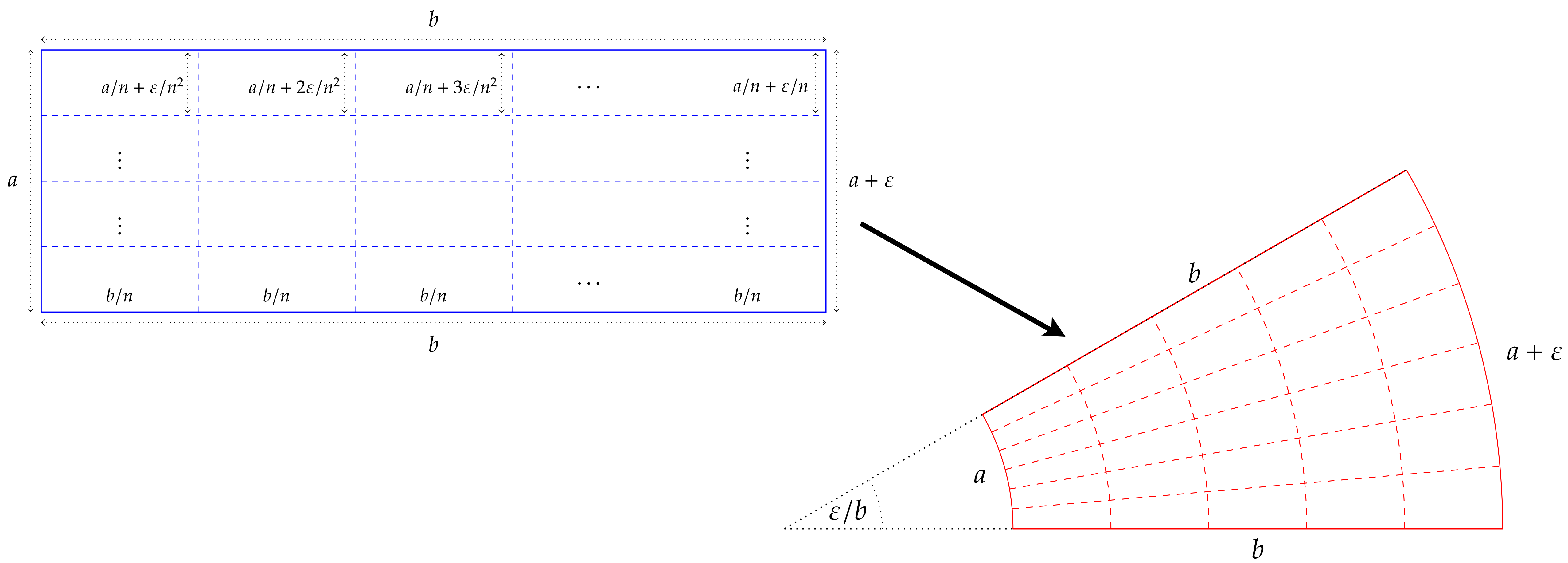
Metric Convergence

Gromov-Hausdorff convergence:

$\mathcal{M}_n \xrightarrow{GH} \mathcal{M}$ if there exist bijections

$$T_n : A_n \subset \mathcal{M}_n \to B_n \subset \mathcal{M}$$

between δ_n-nets A_n and B_n ($\delta_n \to 0$) such that



Metric Convergence

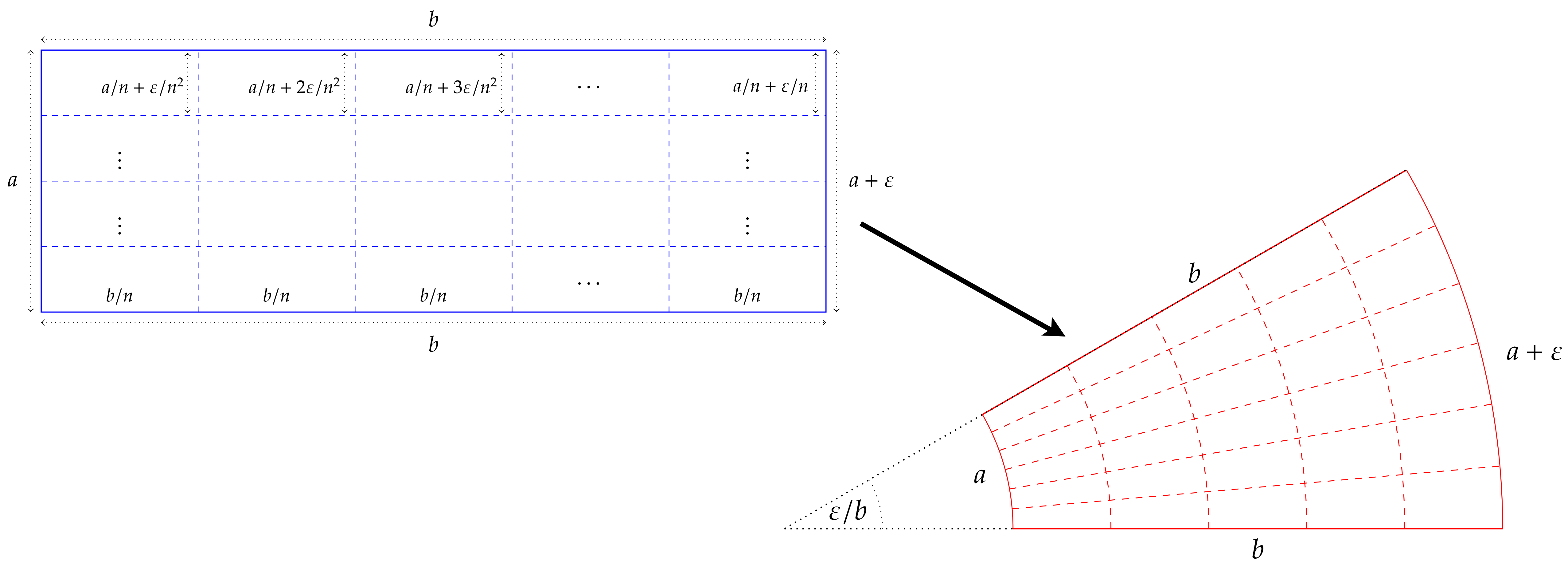
Gromov-Hausdorff convergence:

$\mathcal{M}_n \xrightarrow{GH} \mathcal{M}$ if there exist bijections

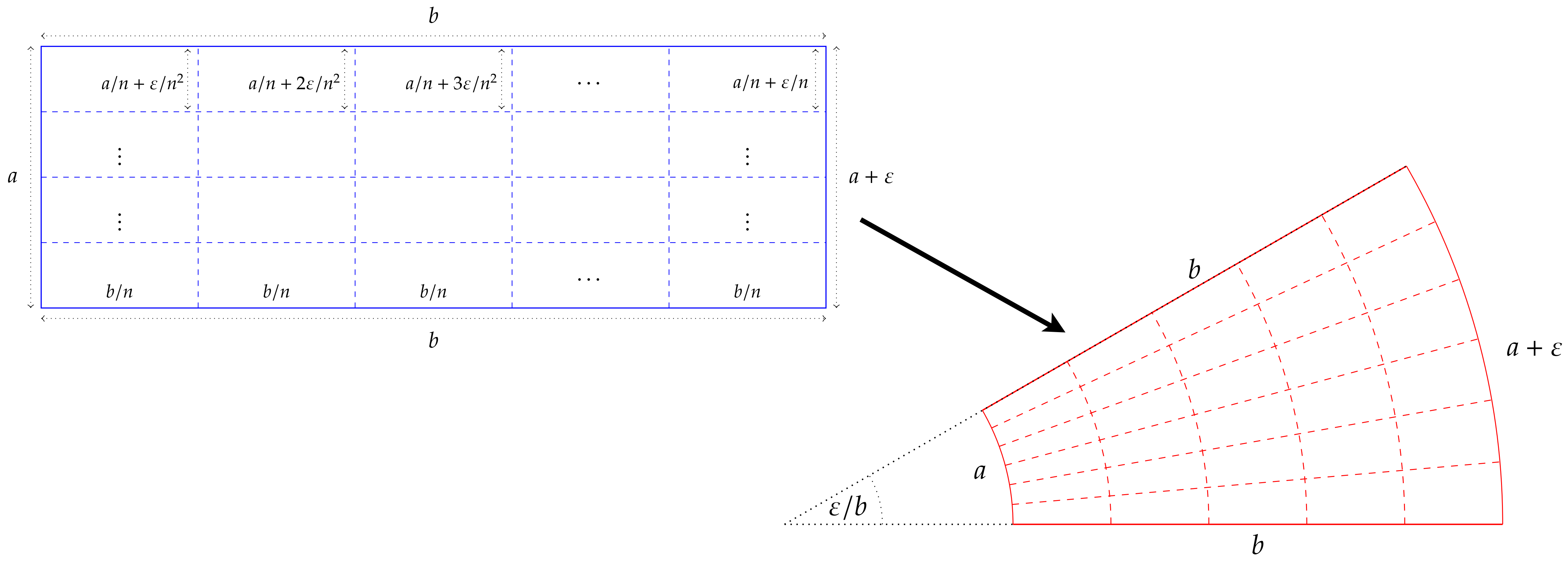
$$T_n : A_n \subset \mathcal{M}_n \to B_n \subset \mathcal{M}$$

between δ_n-nets A_n and B_n ($\delta_n \to 0$) such that

$$\operatorname{dis} T_n = \sup_{x,y \in A_n} |d_{\mathcal{M}_n}(x,y) - d_{\mathcal{M}}(T_n(x), T_n(y))| \to_{n\to\infty} 0$$

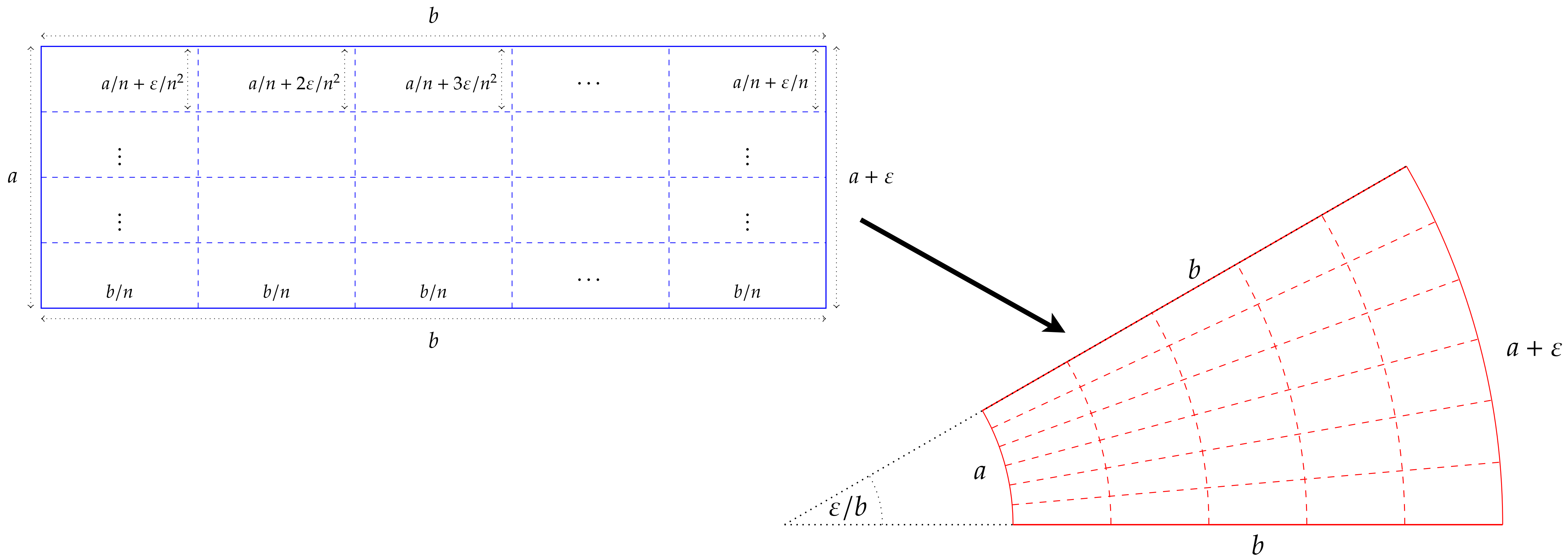


What else converges?



What else converges?

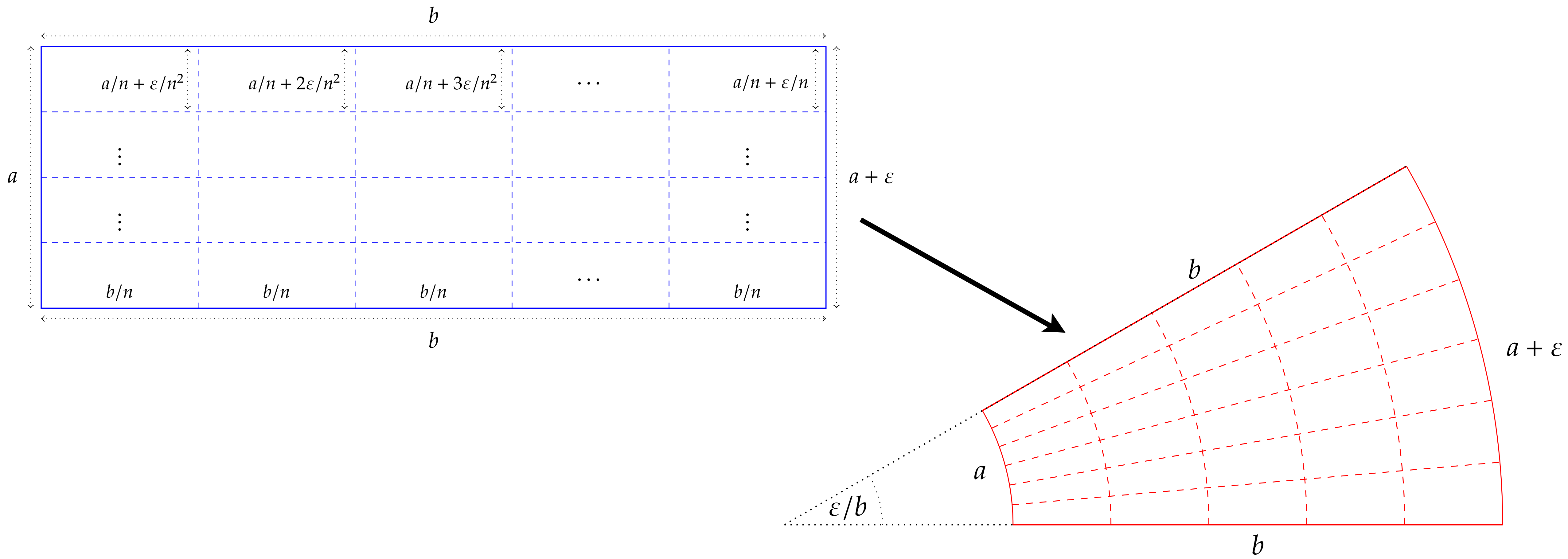
- A_n consists of geodesics (straight lines).



What else converges?

- A_n consists of geodesics (straight lines).

- B_n does not.

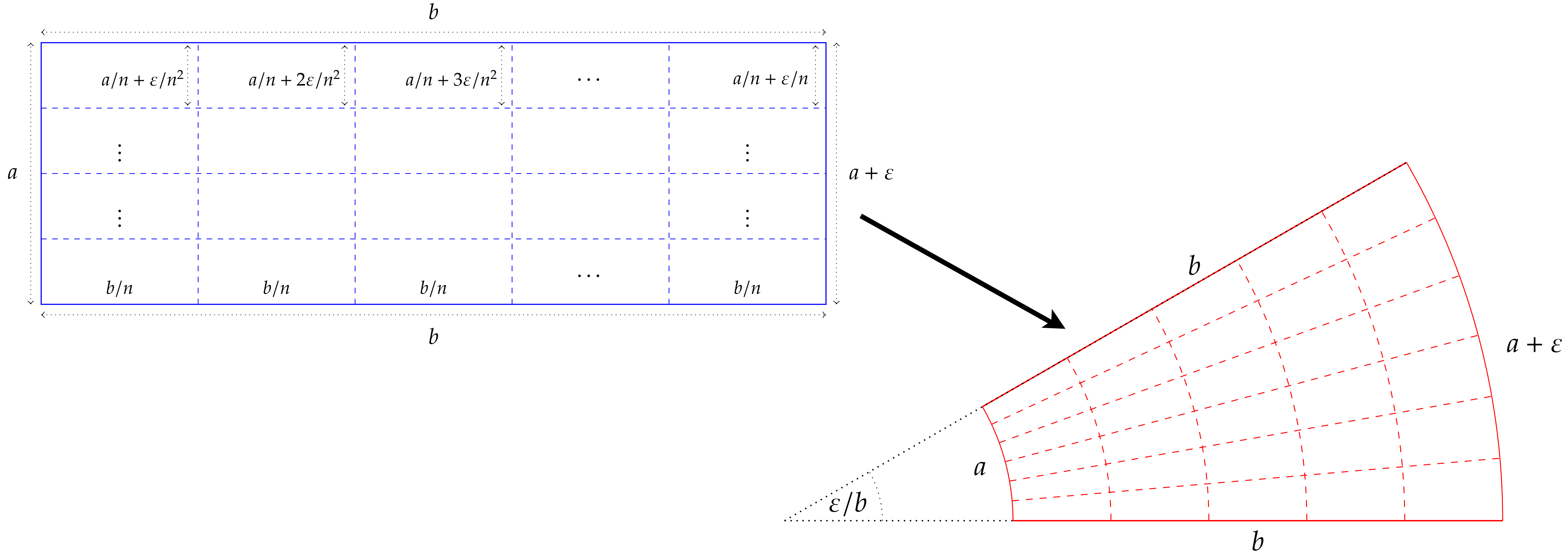


What else converges?

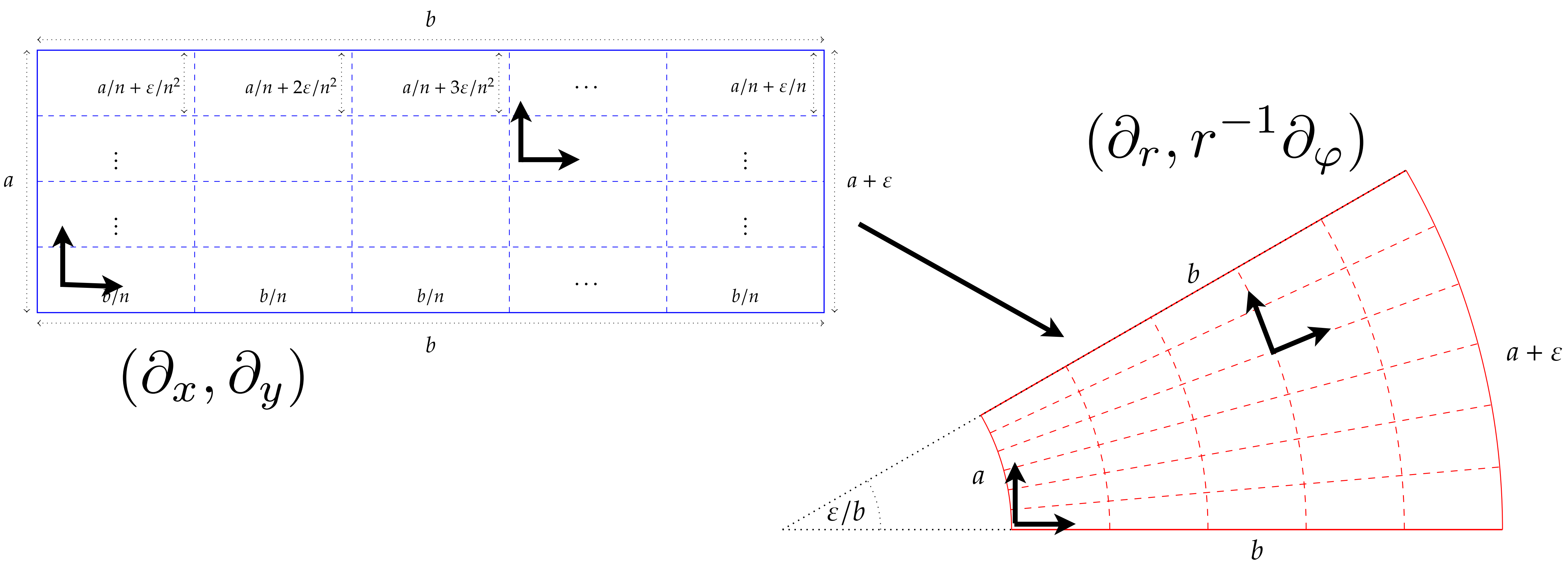
- A_n consists of geodesics (straight lines).

- B_n does not.

  Or does it?

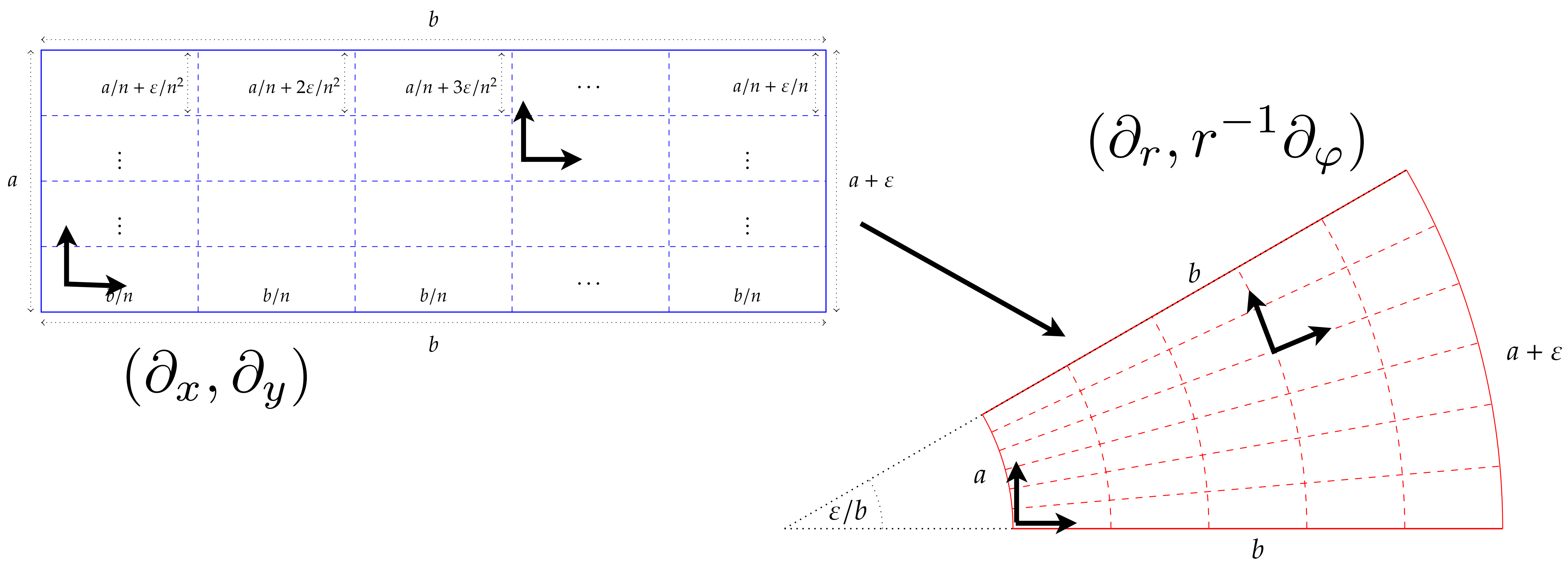


What else converges?



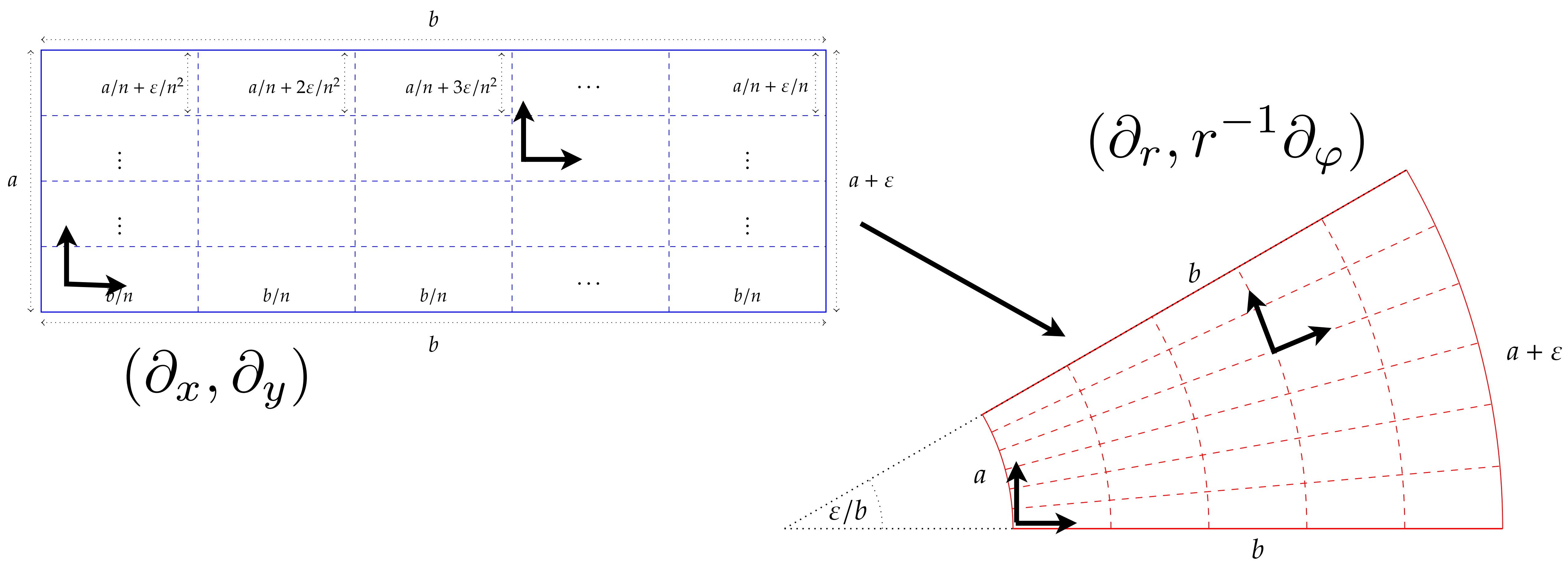
What else converges?

- A_n consists of geodesics w.r.t. the canonical (Levi-Civita) parallel-transport on M_n.



What else converges?

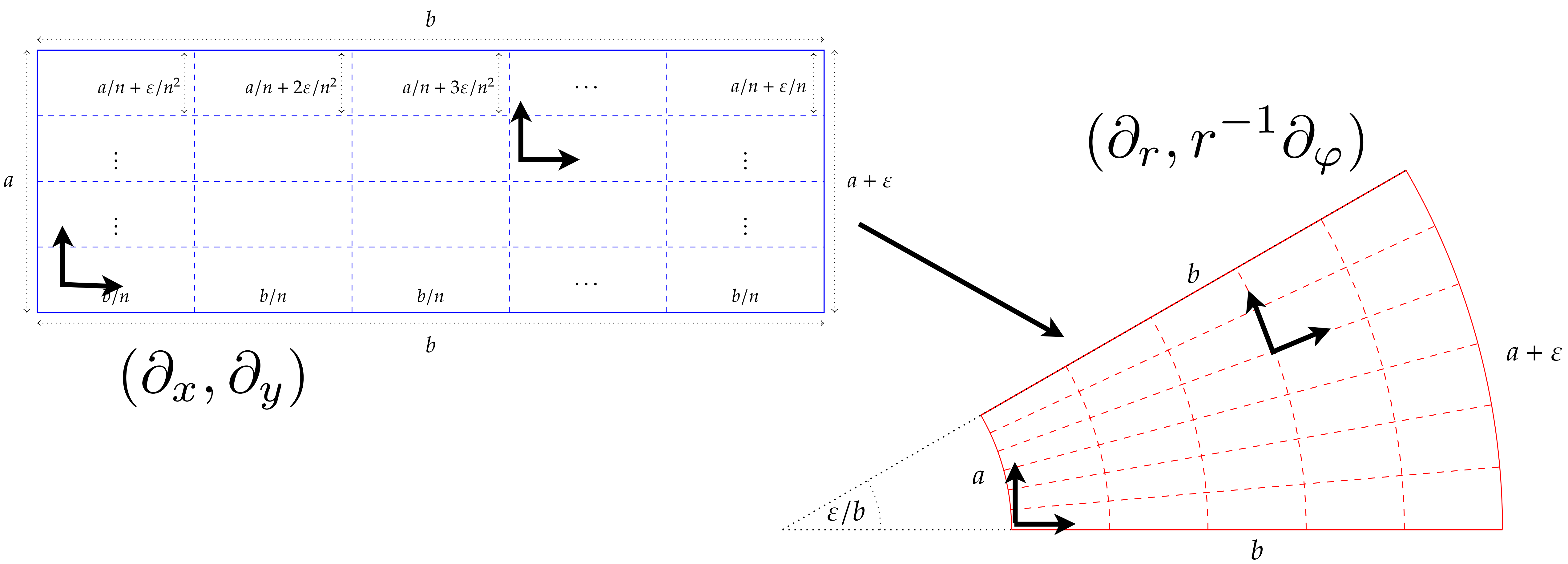
- A_n consists of geodesics w.r.t. the canonical (Levi-Civita) parallel-transport on M_n.
- B_n consists of geodesics w.r.t. a non-canonical one (i.e. with **torsion**) — ∂r and $r^{-1}\partial\varphi$ are parallel.



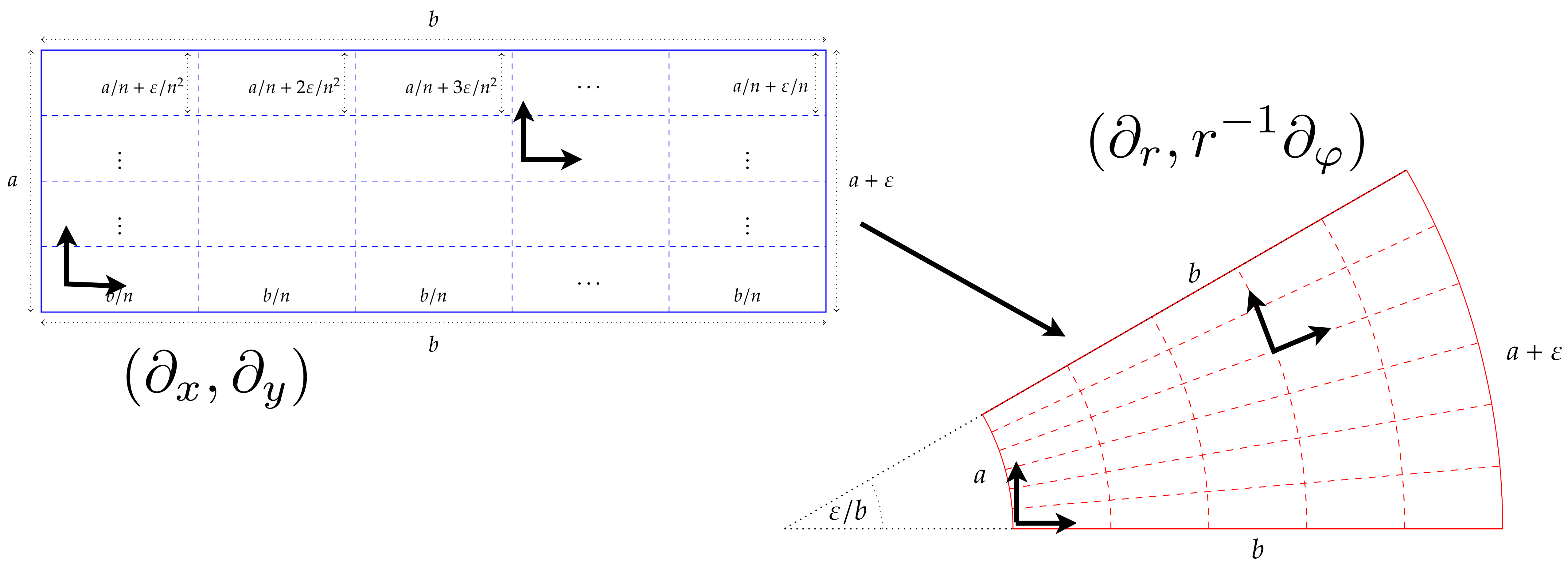
What else converges?

- A_n consists of geodesics w.r.t. the canonical (Levi-Civita) parallel-transport on M_n.
- B_n consists of geodesics w.r.t. a non-canonical one (i.e. with **torsion**) — ∂r and $r^{-1}\partial\varphi$ are parallel.

Do we have convergence of the parallel-transport?

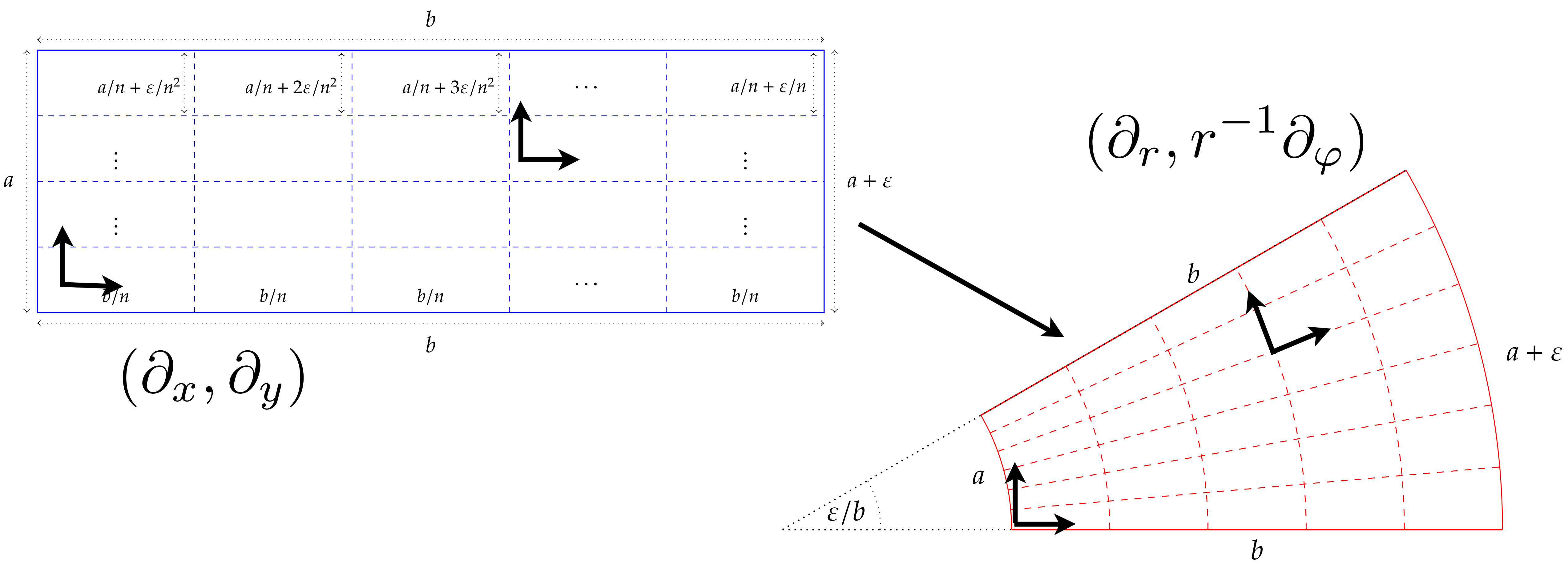


Convergence of the parallel-transport



Convergence of the parallel-transport

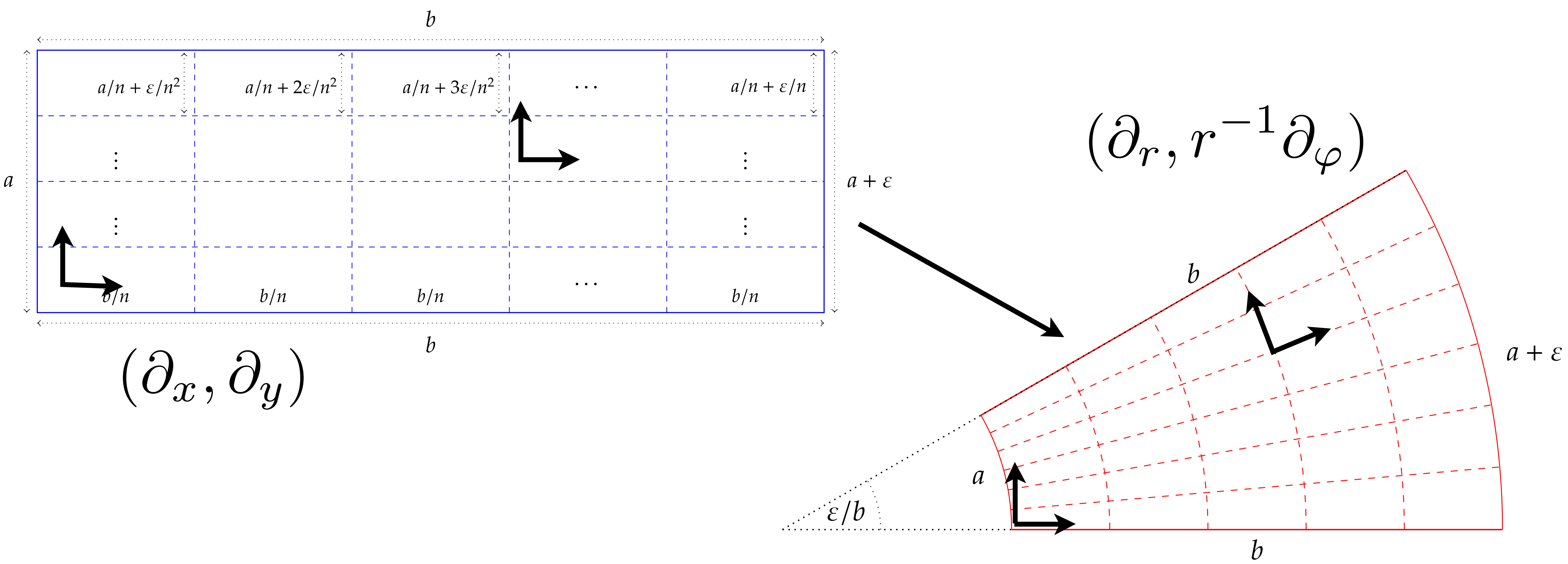
$T_n : A_n \to B_n$ can be extended to a smooth embedding



Convergence of the parallel-transport

$T_n\colon A_n \to B_n$ can be extended to a smooth embedding

$$F_n : \mathcal{M}_n \to \mathcal{M}$$

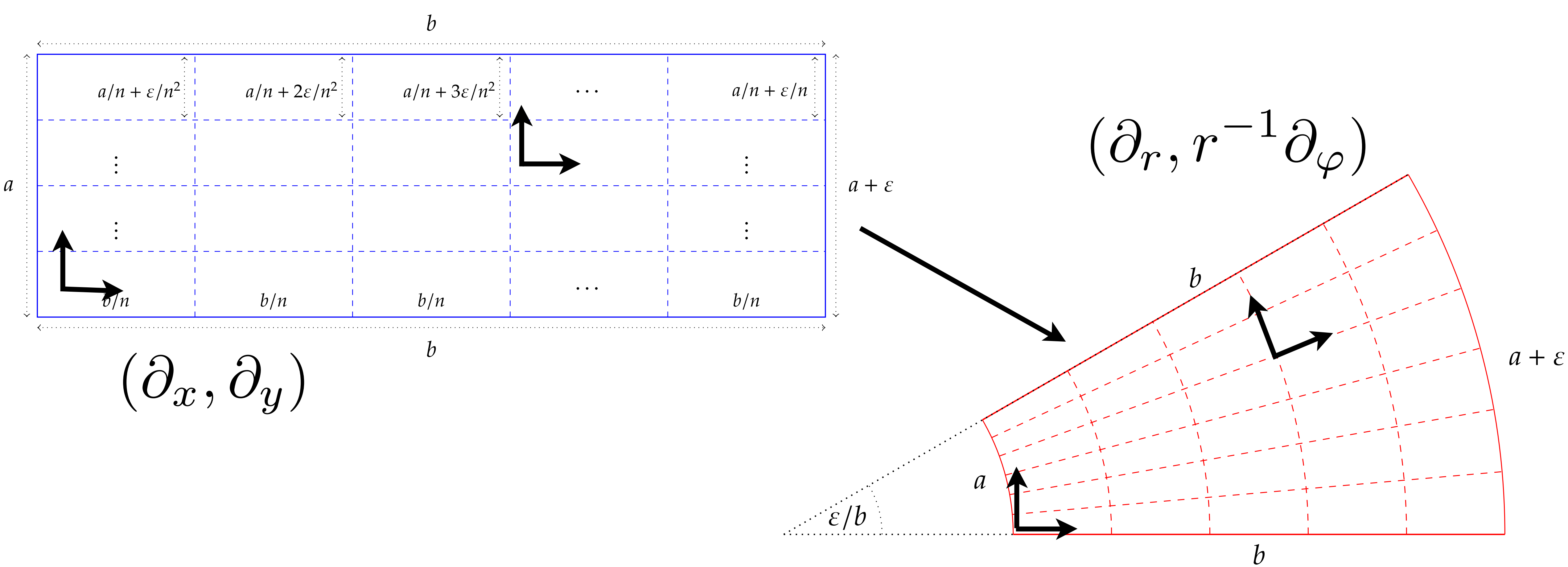


Convergence of the parallel-transport

$T_n\colon A_n \to B_n$ can be extended to a smooth embedding

$$F_n : \mathcal{M}_n \to \mathcal{M}$$

Parallel-transport operators converge:



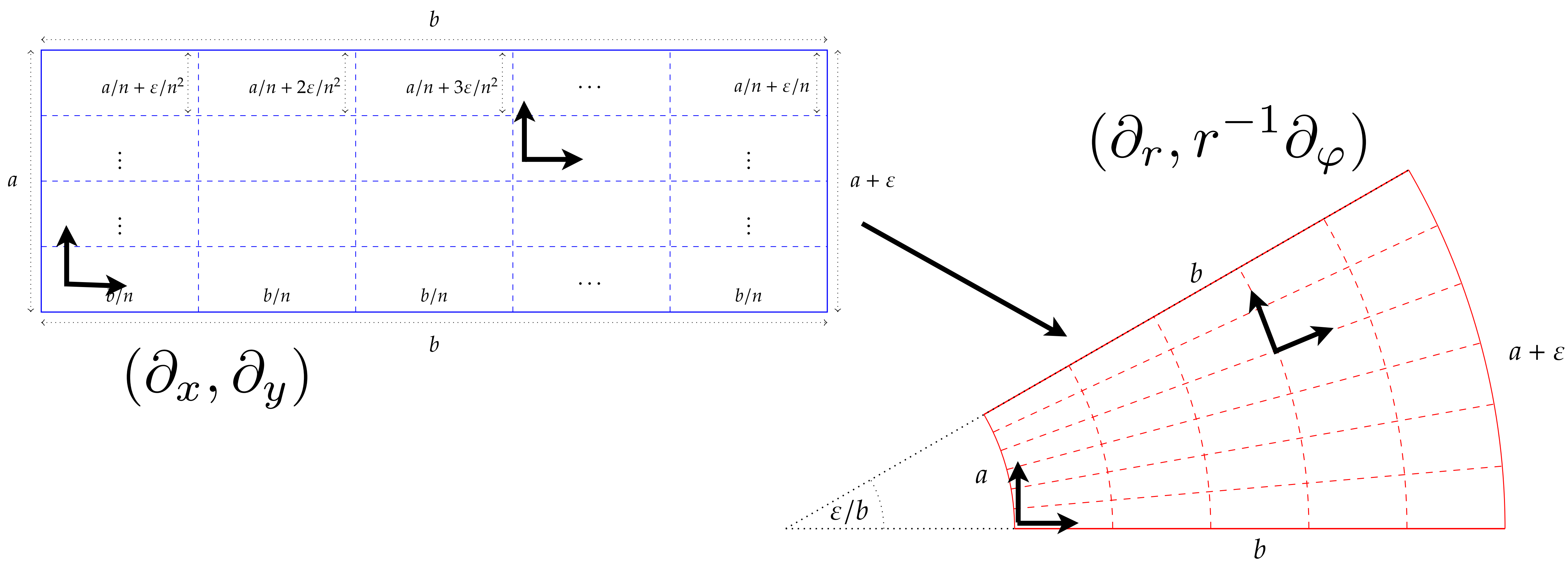
Convergence of the parallel-transport

$T_n\colon A_n \to B_n$ can be extended to a smooth embedding

$$F_n : \mathcal{M}_n \to \mathcal{M}$$

Parallel-transport operators converge:

$$\lim_{n\to\infty} \int_{F_n(\mathcal{M}_n)} |dF_n(\partial_x, \partial_y) - (\partial_r, r^{-1}\partial_\varphi)| = 0$$



Convergence of the parallel-transport

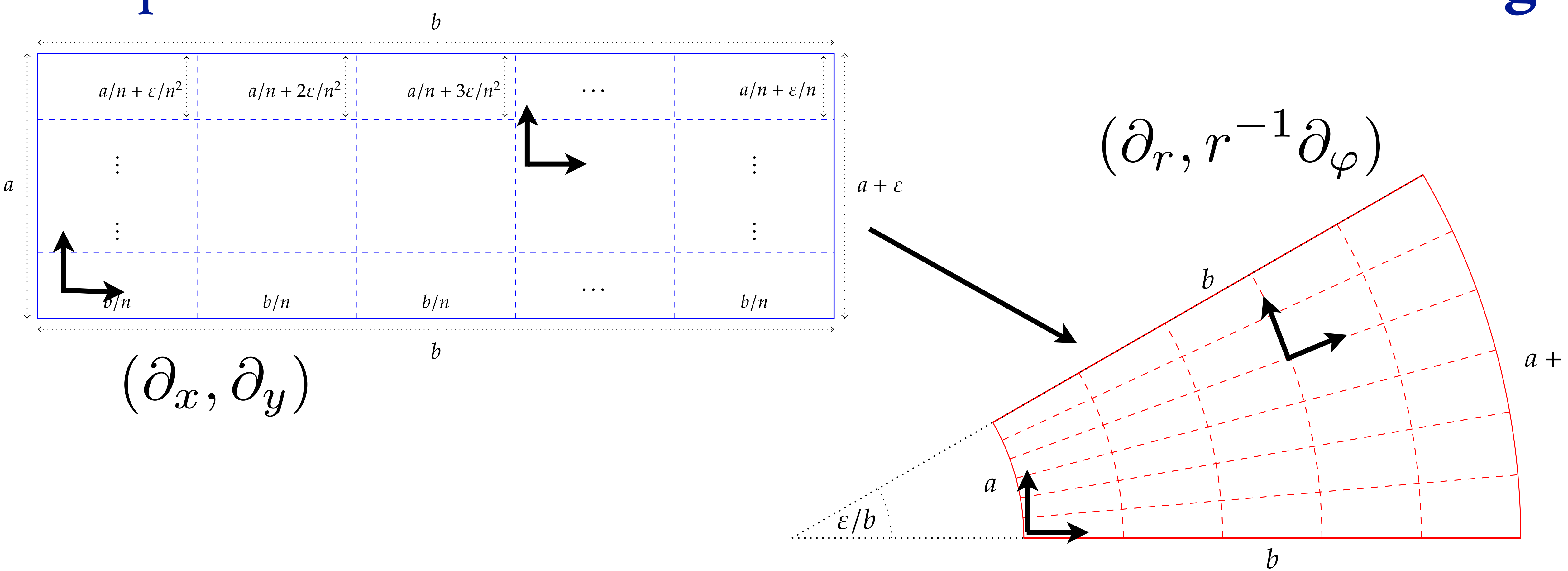
$T_n\colon A_n \to B_n$ can be extended to a smooth embedding

$$F_n : \mathcal{M}_n \to \mathcal{M}$$

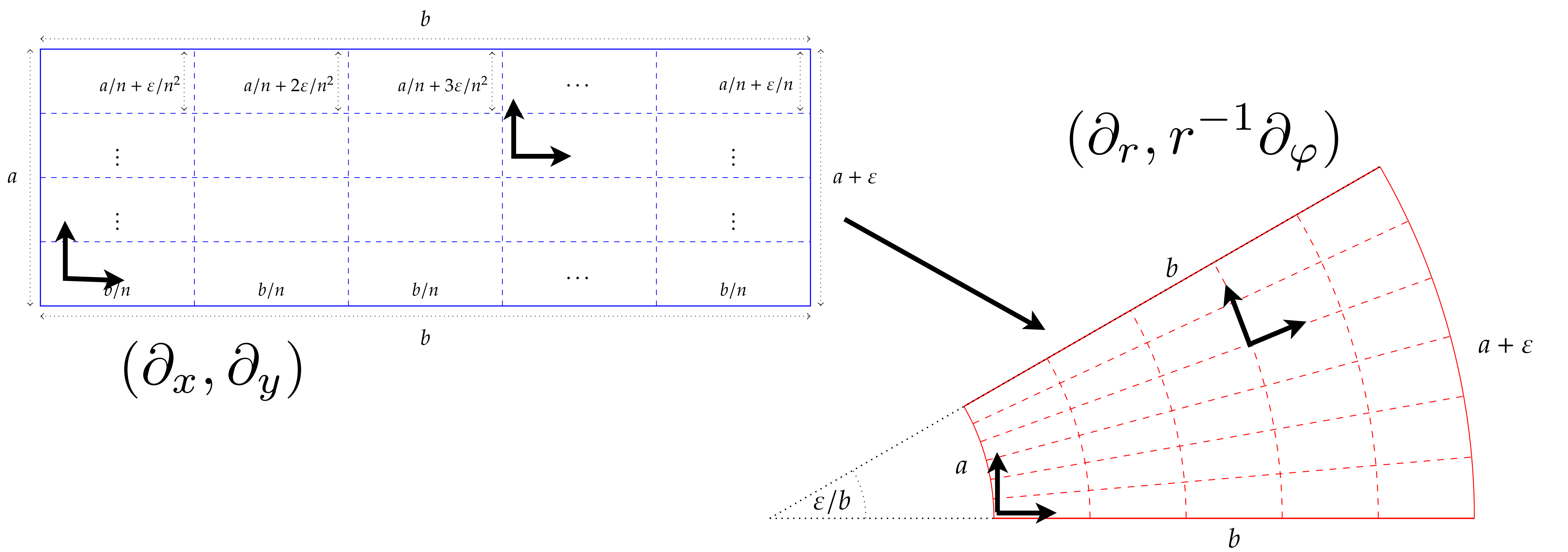
Parallel-transport operators converge:

$$\lim_{n\to\infty} \int_{F_n(\mathcal{M}_n)} |dF_n(\partial_x, \partial_y) - (\partial_r, r^{-1}\partial_\varphi)| = 0$$

Components of the **covariant derivative do not converge**!

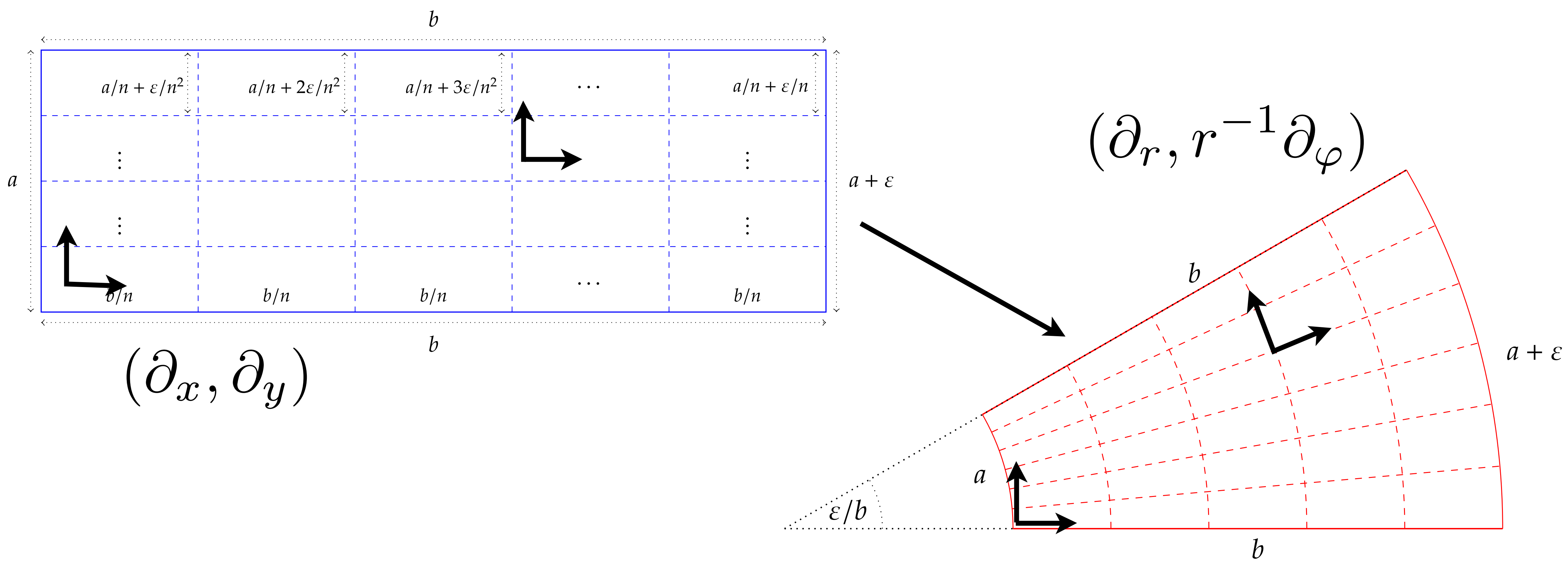


Uniqueness



Uniqueness

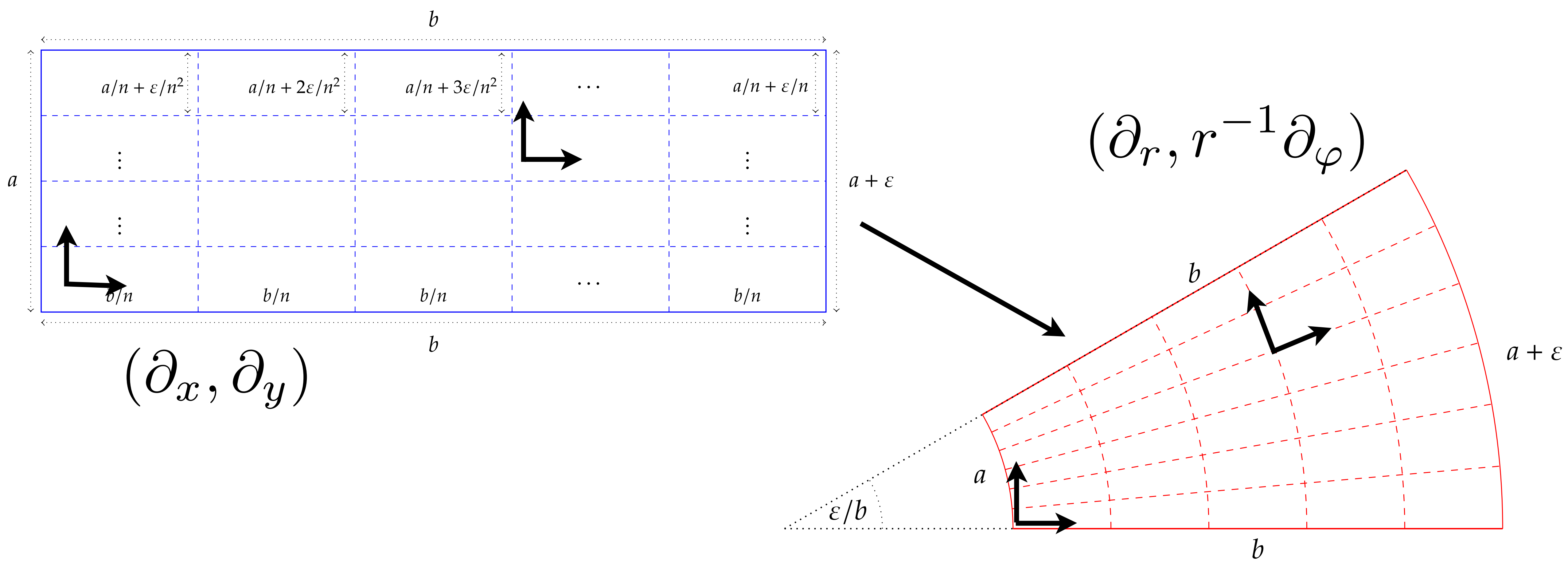
Metric limit: unique by properties of GH convergence.



Uniqueness

Metric limit: unique by properties of GH convergence.

Is the limit parallel-transport well-defined?
Does it depend on the choice of the embeddings F_n and the parallel frame fields $(\partial x, \partial y)$?



Uniqueness

Theorem: Let (M_n, g_n, Π_n), (M, g, Π) be manifolds endowed with path-independent parallel-transport operators (equiv. flat cov. derivatives) such that there exist embeddings $F_n : \mathcal{M}_n \to \mathcal{M}$ such that:

Uniqueness

Theorem: Let (M_n, g_n, Π_n), (M, g, Π) be manifolds endowed with path-independent parallel-transport operators (equiv. flat cov. derivatives) such that there exist embeddings $F_n : \mathcal{M}_n \to \mathcal{M}$ such that:

1. F_n are asymptotically surjective.

Uniqueness

Theorem: Let (M_n, g_n, Π_n), (M, g, Π) be manifolds endowed with path-independent parallel-transport operators (equiv. flat cov. derivatives) such that there exist embeddings $F_n : \mathcal{M}_n \to \mathcal{M}$ such that:

1. F_n are asymptotically surjective.
2. The distortion of F_n tends to zero.

Uniqueness

Theorem: Let (M_n, g_n, Π_n), (M, g, Π) be manifolds endowed with path-independent parallel-transport operators (equiv. flat cov. derivatives) such that there exist embeddings $F_n : \mathcal{M}_n \to \mathcal{M}$ such that:

1. F_n are asymptotically surjective.
2. The distortion of F_n tends to zero.
3. F_n are asymptotically rigid on the mean.

Uniqueness

Theorem: Let (M_n, g_n, Π_n), (M, g, Π) be manifolds endowed with path-independent parallel-transport operators (equiv. flat cov. derivatives) such that there exist embeddings $F_n : \mathcal{M}_n \to \mathcal{M}$ such that:

1. F_n are asymptotically surjective.
2. The distortion of F_n tends to zero.
3. F_n are asymptotically rigid on the mean.
4. There exist Π_n-parallel frame fields E_n and a Π-parallel frame field E such that

Uniqueness

Theorem: Let (M_n,g_n,Π_n), (M,g,Π) be manifolds endowed with path-independent parallel-transport operators (equiv. flat cov. derivatives) such that there exist embeddings $F_n : \mathcal{M}_n \to \mathcal{M}$ such that:

1. F_n are asymptotically surjective.
2. The distortion of F_n tends to zero.
3. F_n are asymptotically rigid on the mean.
4. There exist Π_n-parallel frame fields E_n and a Π-parallel frame field E such that

$$\lim_{n\to\infty} \int_{F_n(\mathcal{M}_n)} |dF_n(E_n) - E| = 0$$

Uniqueness

Theorem: Let *(M_n,g_n,Π_n)*, *(M,g,Π)* be manifolds endowed with path-independent parallel-transport operators (equiv. flat cov. derivatives) such that there exist embeddings $F_n : \mathcal{M}_n \to \mathcal{M}$ such that:

1. F_n are asymptotically surjective.
2. The distortion of F_n tends to zero.
3. F_n are asymptotically rigid on the mean.
4. There exist Π_n-parallel frame fields E_n and a Π-parallel frame field E such that

$$\lim_{n\to\infty} \int_{F_n(\mathcal{M}_n)} |dF_n(E_n) - E| = 0$$

Then *(M,g,Π)* is defined uniquely, that is, independent of the choice of embeddings and frame fields.

What else can we construct?

What else can we construct?

- Essentially, any compact 2-manifold with boundaries endowed with a (metric) path-independent parallel-transport.

What else can we construct?

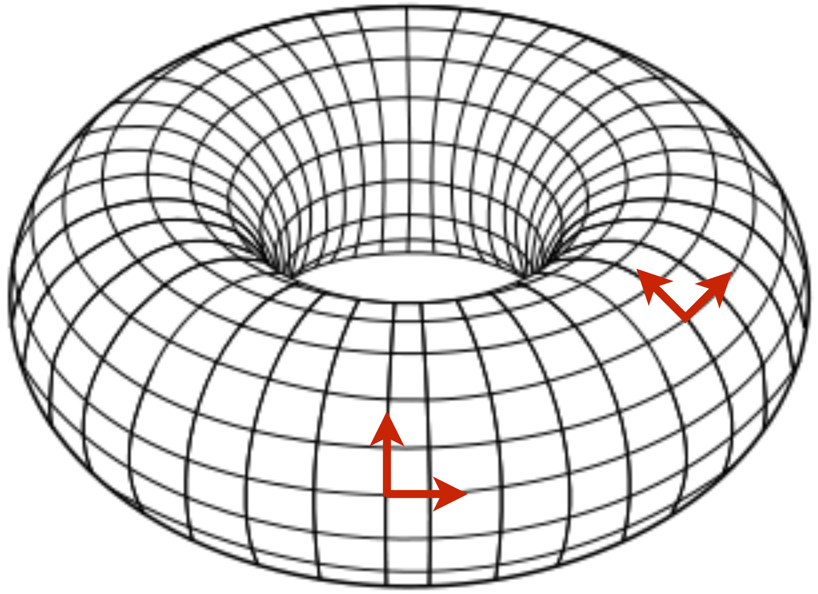
- Essentially, any compact 2-manifold with boundaries endowed with a (metric) path-independent parallel-transport.
- Even if the manifold itself is non-flat!

What else can we construct?

- Essentially, any compact 2-manifold with boundaries endowed with a (metric) path-independent parallel-transport.
- Even if the manifold itself is non-flat!
- For example, the torus with the direction of the meridians and the parallels as a parallel frame-field.

What else can we construct?

- Essentially, any compact 2-manifold with boundaries endowed with a (metric) path-independent parallel-transport.
- Even if the manifold itself is non-flat!
- For example, the torus with the direction of the meridians and the parallels as a parallel frame-field.



Conclusion

Conclusion

- We proved that a sequence of manifolds with edge-dislocations converges, as the dislocations get denser, to a non-singular manifold.

Conclusion

- We proved that a sequence of manifolds with edge-dislocations converges, as the dislocations get denser, to a non-singular manifold.
- The sequence is endowed with the canonical parallel-transports (equiv. affine connections), which converge to a non-canonical parallel-transport of the limit manifold.

Conclusion

- We proved that a sequence of manifolds with edge-dislocations converges, as the dislocations get denser, to a non-singular manifold.
- The sequence is endowed with the canonical parallel-transports (equiv. affine connections), which converge to a non-canonical parallel-transport of the limit manifold.
- The limit manifold is therefore a manifold with a parallel-transport that carries *torsion*.

Conclusion

- We proved that a sequence of manifolds with edge-dislocations converges, as the dislocations get denser, to a non-singular manifold.
- The sequence is endowed with the canonical parallel-transports (equiv. affine connections), which converge to a non-canonical parallel-transport of the limit manifold.
- The limit manifold is therefore a manifold with a parallel-transport that carries *torsion*.
- This fits the phenomenological description of a continuous distribution of dislocations.

Conclusion

- We proved that a sequence of manifolds with edge-dislocations converges, as the dislocations get denser, to a non-singular manifold.
- The sequence is endowed with the canonical parallel-transports (equiv. affine connections), which converge to a non-canonical parallel-transport of the limit manifold.
- The limit manifold is therefore a manifold with a parallel-transport that carries *torsion*.
- This fits the phenomenological description of a continuous distribution of dislocations.

Thank you for your attention!