Structure of the Excitation Spectrum for Many-Body Quantum Systems

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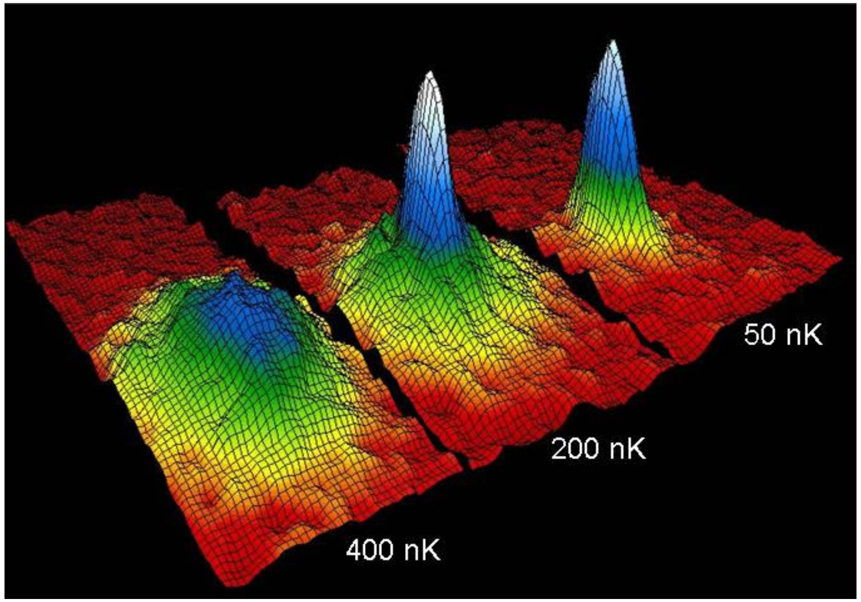
Introduction

First realization of **Bose-Einstein Condensation** (BEC) in cold atomic gases in 1995:

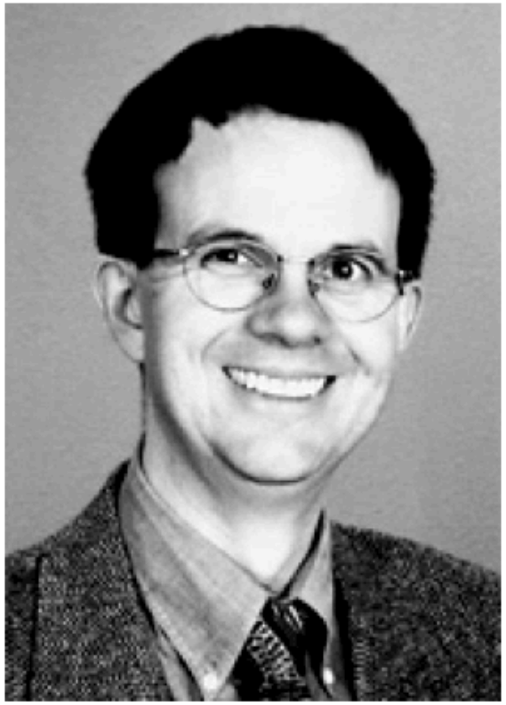


In these experiments, a large number of (bosonic) atoms is confined to a trap and cooled to very low temperatures. Below a **critical temperature** condensation of a large fraction of particles into the same one-particle state occurs.

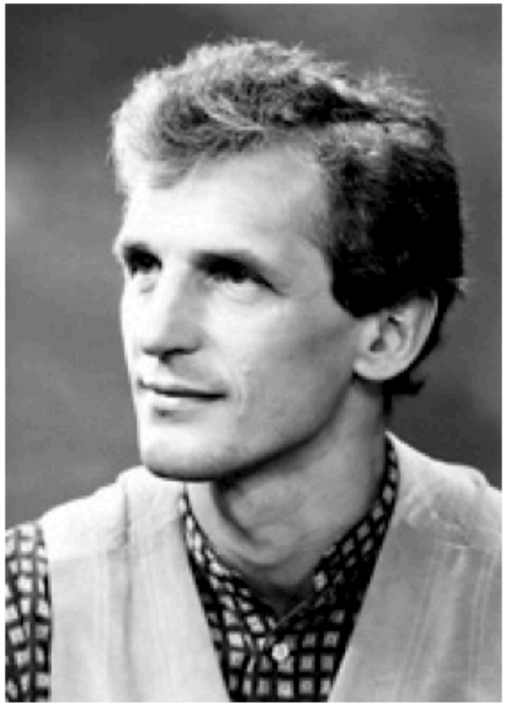
Interesting **quantum phenomena** arise, like the appearance of quantized vortices and superfluidity. The latter is related to the low-energy **excitation spectrum** of the system.

BEC was predicted by Einstein in 1924 from considerations of the **non-interacting** Bose gas. The presence of particle interactions represents a major difficulty for a rigorous derivation of this phenomenon.

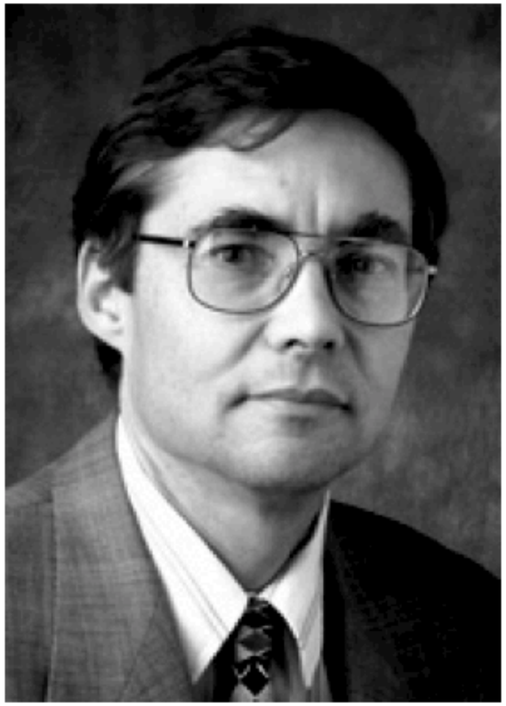
The Nobel Prize in Physics 2001
Eric A. Cornell, Wolfgang Ketterle, Carl E. Wieman



Eric A. Cornell



Wolfgang Ketterle



Carl E. Wieman

The Nobel Prize in Physics 2001 was awarded jointly to Eric A. Cornell, Wolfgang Ketterle and Carl E. Wieman *"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"*.

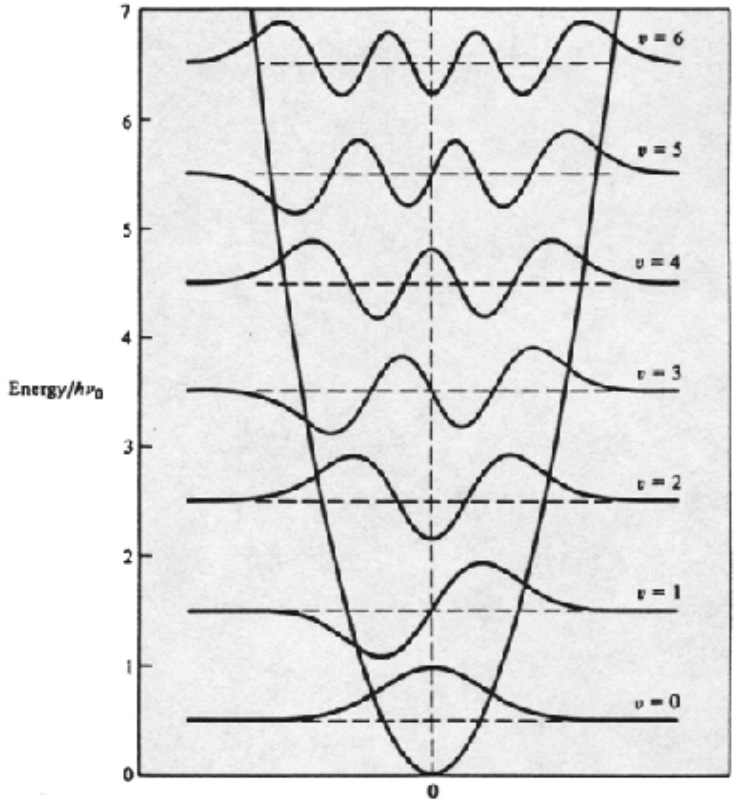
Quantum Mechanics 101

At **low temperature**, quantum mechanics determines the motion of the particles.

Allowed quantum states ψ_j determined by **Schrödinger's equation**

$$-\Delta\psi_j(x) + V(x)\psi_j(x) = E_j\psi_j(x)$$

with $\Delta = \sum_{i=1}^{3} \left(\partial/\partial x^{(i)}\right)^2$. Mathematically extremely well understood. Explicit solutions for some potentials $V(x)$, e.g., harmonic oscillator $V(x) = |x|^2$.



Bosons and Fermions

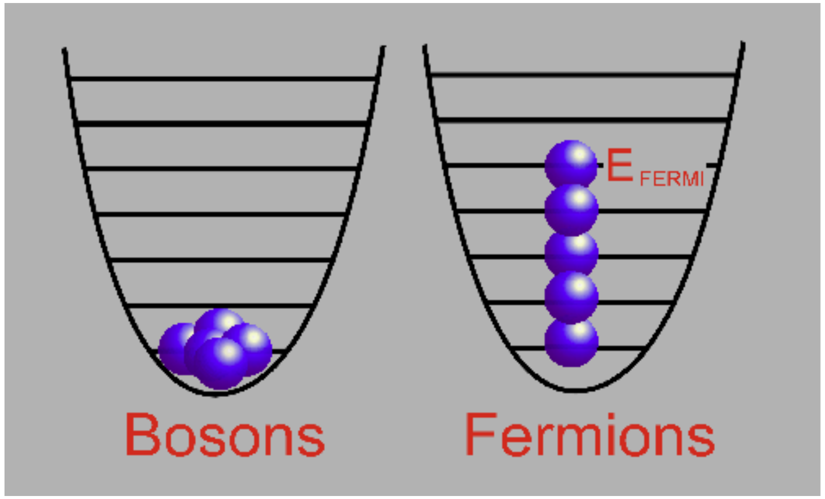
Indistinguishable particles in nature come in two types: **bosons (fermions)** have permutation-(anti-)symmetric wavefunctions

$$\Psi(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) = \underbrace{(-1)}_{\text{for fermions}} \Psi(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N)$$

If one **neglects interactions** among the particles, $\Psi(x_1,\ldots,x_N)$ is just an (anti-) symmetrized product of functions

$$\psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_N}(x_N)$$

with ψ_k appearing n_k times, say. For fermions, $n_k \in \{0,1\}$ (**Pauli exclusion principle**), for bosons $n_k \in \{0,1,\ldots,N\}$.



Bosons at zero temperature display complete **Bose-Einstein condensation**.

The Bose Gas: A Quantum Many-Body Problem

Quantum-mechanical description in terms of the **Hamiltonian** for a gas of N bosons with pair-interaction potential $v(x)$. In appropriate units,

$$\boxed{H_N = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \leq i < j \leq N} v(x_i - x_j)}$$

The kinetic energy is described by the Δ, the Laplacian on a box $[0, L]^3$, with periodic boundary conditions.

As appropriate for **bosons**, H acts on **permutation-symmetric** wave functions $\Psi(x_1, \ldots, x_N)$ in $\bigotimes^N L^2([0, L]^3)$.

The interaction v is assumed to be **repulsive** and of **short range**.

Example: hard spheres, $v(x) = \infty$ for $|x| \leq a$, 0 for $|x| > a$.

Quantities of Interest

- **Ground state energy**

$$E_0(N,L) = \inf \operatorname{spec} H_N$$

In particular, energy density in the **thermodynamic limit** $N \to \infty$, $L \to \infty$ with $N/L^3 = \varrho$ fixed, i.e.,

$$e(\varrho) = \lim_{L\to\infty} \frac{E_0(\varrho L^3, L)}{L^3}$$

- At **positive temperature** $T = \beta^{-1} > 0$, one looks at the **free energy**

$$F(N,L,T) = -\frac{1}{\beta} \ln \operatorname{Tr} \exp(-\beta H_N)$$

and the corresponding energy density in the thermodynamic limit

$$f(\varrho,T) = \lim_{L\to\infty} \frac{F(\varrho L^3, L, T)}{L^3}$$

- The **one-particle density matrix** of the ground state Ψ_0 (or any other state) is given by the integral kernel

$$\gamma_0(x,x') = N \int_{\mathbb{R}^{3(N-1)}} \Psi_0(x,x_2,\ldots,x_N)\Psi_0^*(x',x_2,\ldots,x_N)\,dx_2\cdots dx_N$$

It satisfies $0 \leq \gamma_0 \leq N$ as an operator, and $\mathrm{Tr}\,\gamma_0 = N$.

Bose-Einstein condensation in a state means that the one-particle density matrix γ_0 has an eigenvalue of order N, i.e., that $\|\gamma_0\|_\infty = O(N)$. The corresponding eigenfunction is called the **condensate wave function**.

For Gibbs states of translation invariant systems

$$\|\gamma_0\|_\infty = \frac{1}{L^3}\int_{[0,L]^6} \gamma_0(x,x')dx\,dx'$$

and this being order $N = \varrho L^3$ means that $\gamma_0(x,x')$ does not decay as $|x-x'| \to \infty$, which is also termed **long range order**.

BEC is expected to occur below a **critical temperature**.

Satyendra Nath Bose
(1894–1974)

Albert Einstein
(1879–1955)

- The structure of the **excitation spectrum**, i.e., the spectrum of H_N above the ground state energy $E_0(N)$, and the relation of the corresponding eigenstates to the ground state.

  For translation invariant systems, H_N commutes with the **total momentum**

  $$P = -i \sum_{j=1}^{N} \nabla_j$$

  and hence one can look at their **joint spectrum**. Of particular relevance is the infimum

  $$E_q(N,L) = \inf \operatorname{spec} H_N \restriction_{P=q}$$

  and one can investigate the limit

  $$e_q(\varrho) = \lim_{L\to\infty} \left(E_q(\varrho L^3, L) - E_0(\varrho L^3, L) \right) \quad \text{for fixed } \varrho \text{ and } q$$

  For interacting systems, one expects a **linear** behavior of $e_q(\varrho)$ for small q.

The Ideal Bose Gas

For **non-interacting bosons** ($v \equiv 0$), the free energy can be calculated explicitly:

$$f_0(\varrho, T) = \sup_{\mu<0} \left[\mu\varrho + \frac{1}{(2\pi)^3\beta} \int_{\mathbb{R}^3} \ln\left(1 - \exp(-\beta(p^2 - \mu))\right) dp \right]$$

If

$$\varrho \geq \varrho_c(\beta) \equiv \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{e^{\beta p^2} - 1} dp = \left(\frac{T}{4\pi}\right)^{3/2} \zeta(3/2)$$

the supremum is achieved at $\mu = 0$ and hence $\partial f_0/\partial\varrho = 0$ for $\varrho \geq \varrho_c$. In other words, the **critical temperature** equals

$$T_c^{(0)}(\varrho) = \frac{4\pi}{\zeta(3/2)^{2/3}}\, \varrho^{2/3}$$

The **one-particle density matrix** for the ideal Bose gas is given by

$$\gamma_0(x,y) = [\varrho - \varrho_c(\beta)]_+ + \sum_{n\geq 0} \frac{e^{\beta\mu_\varrho n}}{(4\pi\beta n)^{3/2}} e^{-|x-y|^2/(4\beta n)}$$

The spectrum of the Laplacian on $[0, L]^3$ with periodic boundary conditions is

$$\sigma(-\Delta) = \left\{ |p|^2 \,:\, p \in \left(\frac{2\pi}{L} \mathbb{Z} \right)^3 \right\}$$

with corresponding eigenfunctions the plane waves $\varphi_p(x) = L^{-3/2} e^{ip\cdot x}$.

Hence the **spectrum of the ideal gas Hamiltonian**

$$H_N^{(0)} = -\sum_{i=1}^{N} \Delta_i$$

is simply

$$\sigma(H_N^{(0)}) = \left\{ \sum_{p \in (\frac{2\pi}{L}\mathbb{Z})^3} |p|^2 n_p \,:\, n_p \in \mathbb{N}_0\,,\ \sum_p n_p = N \right\}$$

and the corresponding eigenfunctions are **symmetrized tensor products** of the φ_p's.

Second Quantization on Fock space

In the following, it will be convenient to regard $\bigotimes_{\rm sym}^N L^2([0,L]^3)$ as a subspace of the bosonic **Fock space**

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \bigotimes_{\rm sym}^{n} L^2([0,L]^3)$$

A basis of $L^2([0,L]^3)$ is given by the plane waves $L^{-3/2}e^{ipx}$ for $p \in (\frac{2\pi}{L}\mathbb{Z})^3$, and we introduce the corresponding **creation and annihilation operators**, satisfying the **CCR**

$$\left[a_p, a_q\right] = \left[a_p^\dagger, a_q^\dagger\right] = 0 \ , \quad \left[a_p, a_q^\dagger\right] = \delta_{p,q}$$

The Hamiltonian H_N is equal to the restriction to the subspace $\bigotimes_{\rm sym}^N L^2([0,L]^3)$ of

$$\mathcal{H} = \sum_p |p|^2 a_p^\dagger a_p + \frac{1}{2L^3}\sum_p \widehat{v}(p) \sum_{q,k} a_{q+p}^\dagger a_{k-p}^\dagger a_k a_q$$

where

$$\widehat{v}(p) = \int_{[0,L]^3} v(x) e^{-ipx} dx$$

denotes the Fourier transform of v.

The Bogoliubov Approximation

At low energy and for weak interactions, one expects Bose-Einstein condensation, meaning that $a_0^\dagger a_0 \sim N$. Hence $p = 0$ plays a special role.

The **Bogoliubov approximation** consists of

- dropping all terms higher than quadratic in $a_p^\dagger$ and a_p for $p \neq 0$.
- replacing $a_0^\dagger$ and a_0 by $\sqrt{N}$

The resulting Hamiltonian is quadratic in the $a_p^\dagger$ and a_p, and equals

$$\mathcal{H}^{\text{Bog}} = \frac{N(N-1)}{2L^3}\widehat{v}(0) + \sum_{p\neq 0}\left(\left(|p|^2 + \varrho\widehat{v}(p)\right)a_p^\dagger a_p + \tfrac{1}{2}\varrho\widehat{v}(p)\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right)\right)$$

with $\varrho = N/L^3$. It can be diagonalized via a **Bogoliubov transformation**.

Bogoliubov Transformation

Let $b_p = \cosh(\alpha_p) a_p + \sinh(\alpha_p) a^\dagger_{-p}$, with

$$\tanh(\alpha_p) = \frac{|p|^2 + \varrho\widehat{v}(p) - \sqrt{|p|^4 + 2|p|^2\varrho\widehat{v}(p)}}{\varrho\widehat{v}(p)}$$

Here, we have to **assume** that $|p|^2 + 2\varrho\widehat{v}(p) \geq 0$ for all p. The b_p and $b_p^\dagger$ again satisfy **CCR**. A simple calculation yields

$$\mathcal{H}^{\mathrm{Bog}} = E_0^{\mathrm{Bog}} + \sum_{p\neq 0} e_p b_p^\dagger b_p$$

where

$$E_0^{\mathrm{Bog}} = \frac{N(N-1)}{2L^3}\widehat{v}(0) - \frac{1}{2}\sum_{p\neq 0}\left(|p|^2 + \varrho\widehat{v}(p) - \sqrt{|p|^4 + 2|p|^2\varrho\widehat{v}(p)}\right)$$

and

$$e_p = \sqrt{|p|^4 + 2|p|^2\varrho\widehat{v}(p)}$$

Consequences of the Bogoliubov Approximation

The Bogoliubov approximation thus yields the ground state energy density

$$e^{\mathrm{Bog}}(\varrho) = \frac{1}{2}\varrho^2 \widehat{v}(0) - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(|p|^2 + \varrho\widehat{v}(p) - \sqrt{|p|^4 + 2|p|^2\varrho\widehat{v}(p)}\right) dp$$

For **small** ϱ, it turns out that

$$e^{\mathrm{Bog}}(\varrho) = \frac{1}{2}\varrho^2\left(\widehat{v}(0) - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{|\widehat{v}(p)|^2}{|p|^2}dp\right) + 4\pi\frac{128}{15\sqrt{\pi}}\left(\frac{\varrho\widehat{v}(0)}{8\pi}\right)^{5/2} + o(\varrho^{5/2})$$

where

$$\frac{128}{15\sqrt{\pi}} = -\sqrt{\frac{8}{\pi^3}}\int_{\mathbb{R}^3}\left(|p|^2 + 1 - \sqrt{|p|^4 + 2|p|^2} - \frac{1}{2|p|^2}\right) dp$$

Since $\widehat{v}(0) - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{|\widehat{v}(p)|^2}{|p|^2}dp$ are the first two terms in the **Born series** for $8\pi a$, the **scattering length** of v, this leads to the prediction

$$e(\varrho) = 4\pi a\varrho^2\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\varrho a^3} + o(\varrho^{1/2})\right) \qquad \text{[Lee, Huang, Yang, 1957]}$$

The Excitation Spectrum in the Bogoliubov Approximation

The spectrum of $\mathcal{H}^{\mathrm{Bog}} - E^{\mathrm{Bog}}$ is obviously given by

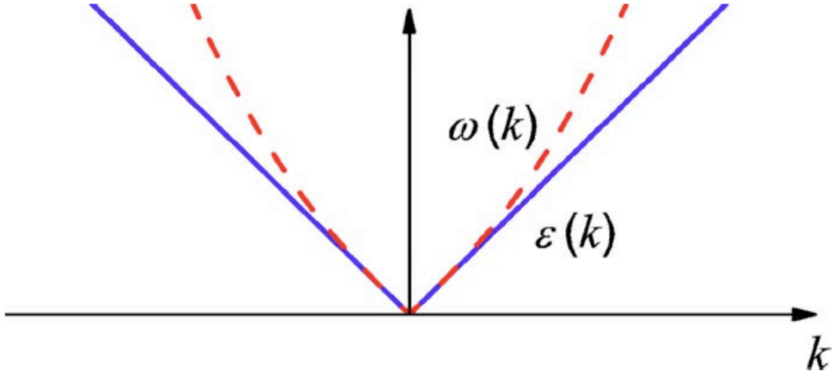
$$\sum_p e_p n_p \quad \text{with } n_p \in \mathbb{N}_0$$

The corresponding eigenstates can be constructed out of the ground state by **elementary excitations**

$$b_{p_n}^\dagger \cdots b_{p_1}^\dagger \Psi_0$$

with $b_p^\dagger = \cosh(\alpha_p) a_p^\dagger + \sinh(\alpha_p) a_{-p}$.

One can also calculate the ground state energy E_q in a sector of total momentum q, and arrives at



$$e_q(\varrho) = \lim_{L\to\infty} \left(E_q^{\mathrm{Bog}} - E_0^{\mathrm{Bog}} \right) = \text{subadditive hull of } e_p = \inf_{\sum_p p n_p = q} \sum_p e_p n_p$$

Validity of the Bogoliubov Approximation

There are only few rigorous results concerning the validity of the Bogoliubov approximation:

- Quite generally, one can show that the **pressure in the thermodynamic limit** is unaffected by the substitution of $a_0^\dagger$ and a_0 (or any other mode) by a c-number [Ginibre 1968; Lieb, Seiringer, Yngvason, 2005; Sütő, 2005]

- The exactly solvable **Lieb-Liniger model** of one-dimensional bosons

$$H_N = \sum_{j=1}^{N} -\frac{\partial^2}{\partial z_j^2} + g \sum_{1\leq i<j\leq N} \delta(z_i - z_j)$$

  on $\bigotimes_{\text{sym}}^N L^2([0,L])$. The Bogoliubov approximation for the ground state energy and the excitation spectrum becomes exact in the weak coupling/high density limit $g/\varrho \to 0$.

Validity of the Bogoliubov Approximation

- For **charged bosons** in a uniform background ("jellium") **Foldy's law**

$$e(\varrho) \approx C\varrho^{5/4}$$

for the ground state energy density has been verified in [Lieb, Solovej, 2001]. Again, the Bogoliubov approximation becomes exact in the high density limit.

- The leading term in the ground state energy of the **low density Bose gas**,

$$e(\varrho) \approx 4\pi a\varrho^2$$

was proved to be correct in [Dyson, 1957] and [Lieb, Yngvason, 1998]. An **upper bound** of the conjectured form

$$4\pi a\varrho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\varrho a^3} + o(\varrho^{1/2})\right)$$

was proved in [Yau, Yin, 2009].

The Bogoliubov Approximation at Low Density

For **small** ϱ, the Bogoliubov approximation can only be strictly valid if

- The third term in the Born series for the scattering length is negligible
- The second term is large compared with $a(a^3\varrho)^{1/2}$.

Consider an interaction potential of the form

$$\frac{a_0}{R^3} v(x/R)$$

for "nice" v with $\int v = 8\pi$, and R a (possibly **density-dependent**) parameter.
The conditions are then

$$\frac{a^3}{R^2} \ll a(a^3\varrho)^{1/2} \ll \frac{a^2}{R}$$

or $a/R \sim (a^3\varrho)^{1/2-\delta}$ with $0 < \delta < 1/4$. Note that $\delta < 1/6$ corresponds to $R \gg \varrho^{-1/3}$.

In [Giuliani, Seiringer, 2009], LHY is proved for small δ. Extension to $\delta < 1/6 + \varepsilon$ in [Lieb, Solovej, in preparation].

The Mean-Field (Hartree) Limit

Consider $L = 1$, for simplicity. The **Hamiltonian** for a gas of N bosons confined to the unit torus $\mathbb{T}^3$, is, in appropriate units,

$$\boxed{H_N = -\sum_{i=1}^{N} \Delta_i + \frac{1}{N-1} \sum_{1 \le i < j \le N} v(x_i - x_j)}$$

The interaction is weak and we write it as $(N-1)^{-1} v(x)$. The case of fixed, N-independent v corresponds to the **mean-field** or **Hartree** limit.

The ground state energy is determined, to leading order, by minimizing over **product states** $\phi(x_1) \cdots \phi(x_N)$. Bogoliubov's theory describes **fluctuations** around such product states.

For our analysis of the excitation spectrum, we assume that $v(x)$ is bounded and of positive type, i.e.,

$$v(x) = \sum_{p \in (2\pi\mathbb{Z})^3} \widehat{v}(p) e^{ip \cdot x} \quad \text{with } \widehat{v}(p) \ge 0 \ \forall p \in (2\pi\mathbb{Z})^3$$

Quantities of Interest

- **Ground State Energy**, given by

$$E_0(N) = \inf \operatorname{spec} H_N$$

For fixed (i.e., N-independent) v, it is easy to see that $E_0(N) = \frac{1}{2} N \widehat{v}(0) + O(1)$. Can one compute the $O(1)$ term?

- **Excitation Spectrum**. What is the spectrum of $H_N - E_0(N)$? Does it converge as $N \to \infty$? Is the **Bogoliubov approximation** valid? The latter predicts a dispersion law for elementary excitations that is **linear** for small momentum.

- **Bose-Einstein condensation**, concerning the largest eigenvalue of the one-particle density matrix

$$\langle f|\gamma|g\rangle = N \int \overline{f(x)\Psi(x, x_2, \ldots, x_N)} g(y)\Psi(y, x_2, \ldots, x_N)\, dx\, dy\, dx_2 \cdots dx_N$$

For fixed v, one easily sees that $\|\gamma\| \geq N - O(1)$ in the ground state.

Main Results

THEOREM 1. *[S, 2011] The* **ground state energy** $E_0(N)$ *of* H_N *equals*

$$\boxed{E_0(N) = \frac{N}{2}\widehat{v}(0) + E^{\mathrm{Bog}} + O(N^{-1/2})}$$

with

$$E^{\mathrm{Bog}} = -\frac{1}{2}\sum_{p\neq 0}\left(|p|^2 + \widehat{v}(p) - \sqrt{|p|^4 + 2|p|^2\widehat{v}(p)}\right) .$$

Moreover, the **excitation spectrum** *of* $H_N - E_0(N)$ *below an energy* ξ *is equal to*

$$\boxed{\sum_{p\in(2\pi\mathbb{Z})^3\setminus\{0\}} e_p\, n_p + O\left(\xi^{3/2}N^{-1/2}\right)}$$

where

$$e_p = \sqrt{|p|^4 + 2|p|^2\widehat{v}(p)}$$

and $n_p \in \{0,1,2,\dots\}$ *for all* $p \neq 0$.

Momentum Dependence

Corollary 1. *Let $E_P(N)$ denote the ground state energy of H_N in the sector of total momentum P. We have*

$$\boxed{E_P(N) - E_0(N) = \min_{\{n_p\},\, \sum_p p\, n_p = P} \sum_{p \neq 0} e_p\, n_p + O\left(|P|^{3/2} N^{-1/2}\right)}$$

In particular,

$$E_P(N) - E_0(N) \geq |P| \min_p \sqrt{2\widehat{v}(p) + |p|^2} + O(|P|^{3/2} N^{-1/2})$$

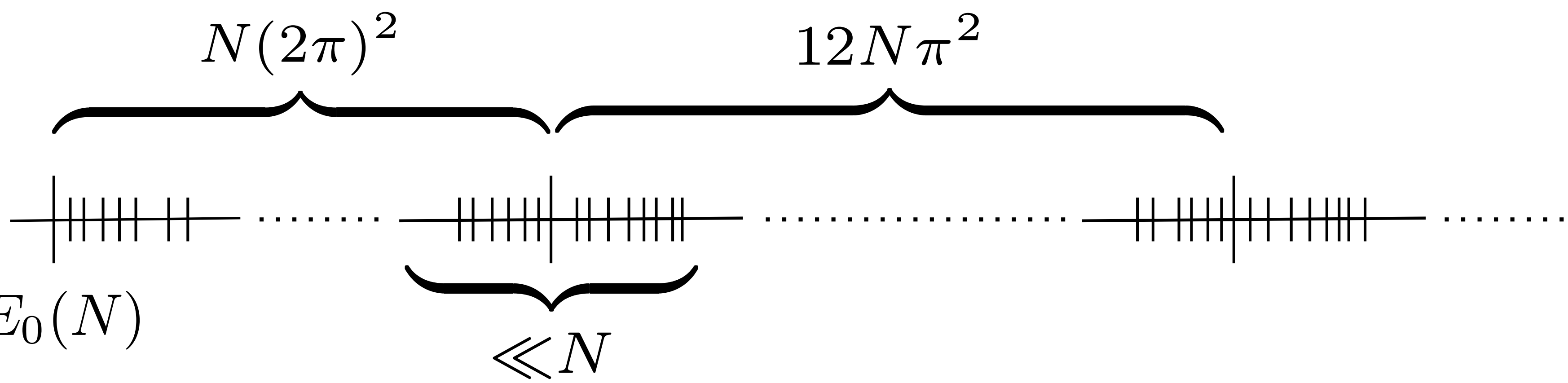
The linear behavior in $|P|$ is important for the **superfluid** behavior of the system. According to Landau, the coefficient in front of $|P|$ is, in fact, the **critical velocity** for frictionless flow.

The Spectrum

Note that under the unitary transformation $U = \exp(-iq \cdot \sum_{j=1}^{N} x_j)$, $q \in (2\pi\mathbb{Z})^3$,

$$U^\dagger H_N U = H_N + N|q|^2 - 2q \cdot P\,,$$

where $P = -i\sum_{j=1}^{N} \nabla_j$ denotes the **total momentum** operator. Hence our results apply equally also to the parts of the spectrum of H_N with excitation energies close to $N|q|^2$, corresponding to **collective excitations** where the particles move uniformly with momentum q.



Generalizations

- **Inhomogeneous systems** in a trap [Grech, Seiringer, 2012], where the condensate is determined by minimizing the **Hartree functional**

$$\int_{\mathbb{R}^3} \left(|\nabla \varphi(x)|^2 + V(x)|\varphi(x)|^2 \right) dx + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\varphi(x)|^2 v(x-y) |\varphi(y)|^2 dx\, dy$$

- More general types of kinetic energy and interaction operators [Lewin, Nam, Serfaty, Solovej, 2013]

- Weakly N-dependent v, scaling to a δ-function as $N \to \infty$ [Dereziński, Napiórkowski, 2013]

- **Collective excitations**, where the condensation occurs in a (non-linear) excited state of the Hartree functional [Nam, Seiringer, 2014]

The Bogoliubov Approximation

In the language of second quantization,

$$\mathcal{H}_N = \sum_{p\in(2\pi\mathbb{Z})^3} |p|^2 a_p^\dagger a_p + \frac{1}{2(N-1)} \sum_p \widehat{v}(p) \sum_{q,k} a_{q+p}^\dagger a_{k-p}^\dagger a_k a_q$$

The **Bogoliubov approximation** consists of

- replacing $a_0^\dagger$ and a_0 by $\sqrt{N}$
- dropping all terms higher than quadratic in $a_p^\dagger$ and a_p, $p \neq 0$.

The resulting quadratic Hamiltonian is $\frac{N}{2}\widehat{v}(0) + \mathcal{H}^{\rm Bog}$, where

$$\mathcal{H}^{\rm Bog} = \sum_{p\neq 0} \left(\left(|p|^2 + \widehat{v}(p)\right) a_p^\dagger a_p + \tfrac{1}{2}\widehat{v}(p) \left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p} \right) \right)$$

It is diagonalized via a **Bogoliubov transformation** $b_p = \cosh(\alpha_p) a_p + \sinh(\alpha_p) a_{-p}^\dagger$, yielding

$$H^{\rm Bog} = E^{\rm Bog} + \sum_{p\neq 0} e_p b_p^\dagger b_p$$

Ideas in the Proof

The proof consists of **two main steps**:

1. Show that H_N is well approximated by an operator similar to the Bogoliubov Hamiltonian $\mathcal{H}^{\mathrm{Bog}}$, but with

$$a_p^\dagger \to c_p^\dagger := \frac{a_p^\dagger a_0}{\sqrt{N}} \quad , \quad a_p \to c_p := \frac{a_p a_0^\dagger}{\sqrt{N}}$$

The resulting operator is quadratic in $c_p^\dagger$ and c_p, and hence particle number conserving.

2. With $d_p = \cosh(\alpha_p) c_p + \sinh(\alpha_p) c_{-p}^\dagger$, analyze the spectrum of

$$\sum_{p \neq 0} e_p d_p^\dagger d_p$$

These do not satisfy CCR anymore, but they do approximately on the subspace where $a_0^\dagger a_0$ is close to N.

Step 1: Approximation by a quadratic Hamiltonian

It is easy to see that

$$N - a_0^\dagger a_0 \leq \text{const.}\ [1 + H_N - E_0(N)]$$

This proves that if the excitation energy is $\ll N$, most particles occupy the zero momentum mode (**Bose-Einstein condensation**).

To show that cubic and quartic terms in $a_p^\dagger$ and a_p, $p \neq 0$, in the Hamiltonian are negligible, one proves a stronger bound of the form

$$\left(N - a_0^\dagger a_0\right)^2 \leq \text{const.}\ \left[1 + (H_N - E_0(N))^2\right]$$

It implies that also the fluctuations in the number of particles outside the condensate are suitably small.

The first statement follows easily from positivity of $\widehat{v}(p)$:

$$\sum_{p\in(2\pi\mathbb{Z})^3\setminus\{0\}} \widehat{v}(p) \left|\sum_{j=1}^{N} e^{ipx_j}\right|^2 \geq 0$$

which can be rewritten as

$$\sum_{1\leq i<j\leq N} v(x_i - x_j) \geq \frac{N^2}{2}\widehat{v}(0) - \frac{N}{2}v(0)$$

Thus

$$H_N \geq -\sum_{i=1}^{N} \Delta_i + \frac{N}{2}\widehat{v}(0) - \frac{N}{2(N-1)} \left(v(0) - \widehat{v}(0)\right) .$$

The statement follows since $-\sum_{i=1}^{N} \Delta_i \geq (2\pi)^2(N - a_0^\dagger a_0)$.

For the second statement one has to work a bit more, and we skip the proof here.

An Algebraic Identity

We conclude that $\mathcal{H}_N$ is, at low energy, well approximated by

$$\frac{N}{2}\widehat{v}(0) + \frac{1}{2}\sum_{p\neq 0}\left[A_p\left(c_p^\dagger c_p + c_{-p}^\dagger c_{-p}\right) + B_p\left(c_p^\dagger c_{-p}^\dagger + c_p c_{-p}\right)\right]$$

with $A_p = |p|^2 + \widehat{v}(p)$ and $B_p = \widehat{v}(p)$. A simple identity (**not using CCR**!) is

$$\begin{aligned}
& A_p\left(c_p^\dagger c_p + c_{-p}^\dagger c_{-p}\right) + B_p\left(c_p^\dagger c_{-p}^\dagger + c_p c_{-p}\right) \\
&= \sqrt{A_p^2 - B_p^2}\left(\frac{\left(c_p^\dagger + \beta_p c_{-p}\right)\left(c_p + \beta_p c_{-p}^\dagger\right)}{1-\beta_p^2} + \frac{\left(c_{-p}^\dagger + \beta_p c_p\right)\left(c_{-p} + \beta_p c_p^\dagger\right)}{1-\beta_p^2}\right) \\
&\quad - \frac{1}{2}\left(A_p - \sqrt{A_p^2 - B_p^2}\right)\left([c_p, c_p^\dagger] + [c_{-p}, c_{-p}^\dagger]\right),
\end{aligned}$$

where

$$\beta_p = \frac{1}{B_p}\left(A_p - \sqrt{A_p^2 - B_p^2}\right) \quad \text{if } B_p > 0\,, \qquad \beta_p = 0 \quad \text{if } B_p = 0.$$

Step 2: The spectrum of $d_p^\dagger d_p$

The usual **Bogoliubov transformation** is of the form

$$e^{-X} a_p e^X = \cosh(\alpha_p) a_p + \sinh(\alpha_p) a_{-p}^\dagger$$

where

$$X = \frac{1}{2} \sum_{p \neq 0} \alpha_p \left(a_p^\dagger a_{-p}^\dagger - a_p a_{-p} \right)$$

This uses the **CCR** $[a_p, a_q^\dagger] = \delta_{p,q}$. Our operators $c_p = a_p a_0^\dagger / \sqrt{N}$ satisfy

$$[c_p, c_q^\dagger] = \delta_{p,q} \frac{a_0 a_0^\dagger}{N} - \frac{a_p a_q^\dagger}{N}$$

which allows us to conclude that

$$e^{-X} a_p e^X = \overbrace{\cosh(\alpha_p) c_p + \sinh(\alpha_p) c_{-p}^\dagger}^{d_p} + \text{Error}$$

with X as before, but with a_p and $a_p^\dagger$ replaced by c_p and $c_p^\dagger$, respectively. Moreover, the error is (relatively) small as long as $(N - a_0^\dagger a_0)^2 \ll N^2$.

Conclusions

- First rigorous results on the **excitation spectrum** of an interacting Bose gas, in a suitable limit of weak, long-range interactions.
- With the notable exception of exactly solvable models in one dimension, this is the only model where rigorous results on the excitation spectrum are available.
- Verification of Bogoliubov's prediction that the spectrum consists of elementary excitations, with energy that is linear in the momentum for small momentum. In particular, **Landau's criterion for superfluidity** is verified.
- **For the future:** more general interactions, less restrictive parameter regime, thermodynamic limit, dilute gas (Gross-Pitaevskii) limit, relation to superfluidity, ...

Open Problems

- Existence of **Bose-Einstein condensation** in the thermodynamic limit
- Correction terms to the energy, validity of the **Lee-Huang-Yang formula** in the low density limit
- Low energy **excitation spectrum** in the thermodynamic limit, and its relation to **superfluidity**
- ...